

WHY DO CRACKS AVOID EACH OTHER ?



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ABSTRACT

The experimentally known phenomenon that originally collinear mode I cracks seem to avoid each other before coalescence is studied by investigating the stability of straight crack paths. To this end a periodic array of approximately collinear but slightly curved cracks is considered. Stress intensity factors for modes I and II are derived and the growth of originally straight cracks under mode I conditions is studied after introduction of a disturbance that forces the cracks to deviate from a straight path. It is shown that the straight crack path is unstable, i.e. that tip to tip coalescence will not take place.

INTRODUCTION

It is an experimental fact that originally collinear cracks, growing under mode I loading, do not follow a straight crack path, but seem to avoid each other, see Fig. 1. Theoretical analysis of this phenomenon will be performed under the assumption that dynamic effects can be disregarded. In order to simplify the analysis a periodic array of approximately collinear cracks is regarded.

A study of curved cracks under arbitrary loading was carried out by Gol'dstein and Salganik [1], [2] and by Gol'dstein and Savova [3]. They regarded the cracks as continuous arrays of dislocations and presented a method for calculating the stress intensity factors. They also gave explicit results for slightly curved cracks. The same results were derived by Cotterell and Rice [4] who used a preturbation method. In both papers an investigation of the crack path at quasi-static growth was carried out.

In the present paper the method introduced by Gol'dstein and Salganik will be used for studying stress intensity factors of a periodic array of slightly curved cracks. Results are given for both modes I and II. Growth of an originally straight crack is studied under mode I conditions after introduction of a small disturbance that forces the crack to deviate from a straight path.

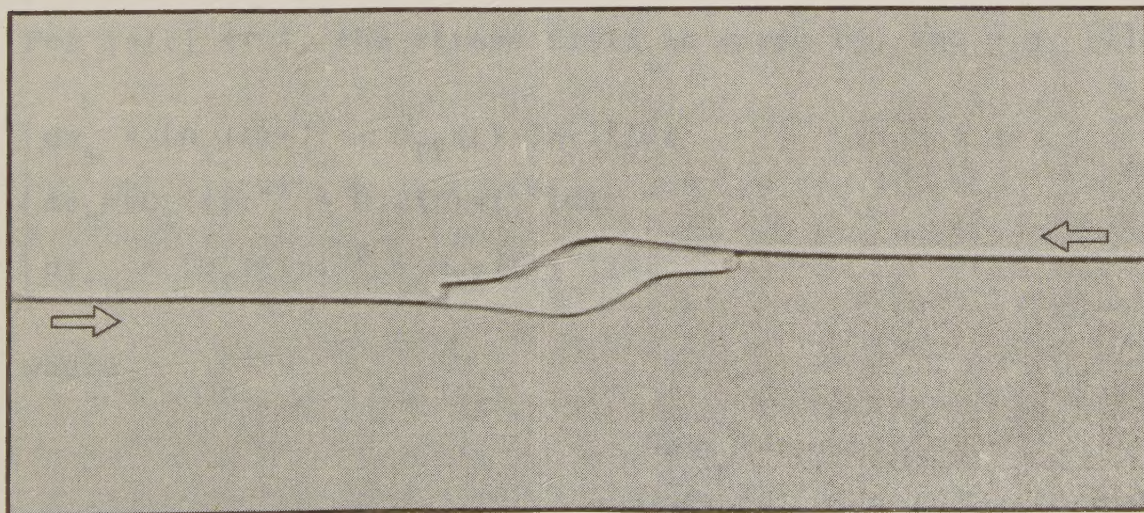


Fig. 1. Paths in a thin plate of polymethylmethacrylate from two edge cracks, originally opposite to each other. Arrows show direction of crack tip movements.

DERIVATION OF STRESS INTENSITY FACTORS FOR SLIGHTLY CURVED CRACKS

Let $y(x)$ describe a smooth line in the xy -plane such that $|y'(x)| \ll 1$ everywhere. Consider an edge dislocation at $(x = \xi, y = y(\xi))$ according to Fig. 2.

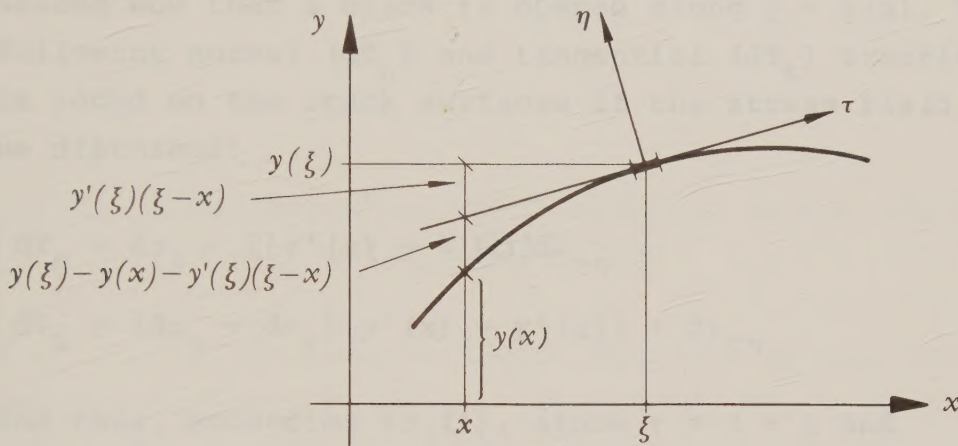


Fig. 2. Global (x, y) and local (τ, η) coordinates for a curved crack and some geometrical relations.

Introduce a new coordinate system (τ, η) such that the dislocation at $x = \xi$ has the infinitesimal Burger's vector $b_I(\xi)d\xi$ in positive η -direction and $b_{II}(\xi)d\xi$ in positive τ -direction.

For $|\eta/\tau| \ll 1$, the stress field is given by, see e.g. [5]:

$$\begin{cases} d\sigma_\tau = [D_I(\xi)\tau^{-1} - D_{II}(\xi) \cdot 3\eta\tau^{-2}]d\xi \\ d\sigma_\eta = [D_I(\xi)\tau^{-1} + D_{II}(\xi)\eta\tau^{-2}]d\xi \\ d\tau_{\tau\eta} = [D_I(\xi)\eta\tau^{-2} + D_{II}(\xi)\tau^{-1}]d\xi \end{cases} \quad (1)$$

where

$$\begin{cases} D_{I,II}(x) = 2G[\pi(\kappa + 1)]^{-1} b_{I,II}(x) \\ b_{I,II}(x) = du_{I,II}(x)/dx \\ u_{I,II}(x) \text{ are the displacements} \\ \kappa = 3-4\nu \text{ in plane strain } (\nu = \text{Poisson's ratio}) \\ \kappa = (3-\nu)(1+\nu)^{-1} \text{ in plane stress} \end{cases}$$

Assume now that a crack is opened along $y = y(x)$. Then the following normal (dT_n) and tangential (dT_t) tractions must be added on the crack surfaces if the stress field shall not be disturbed:

$$\begin{cases} dT_n = d\sigma_\eta - 2[y'(x) - y'(\xi)]d\tau_{\tau\eta} \\ dT_t = (d\sigma_\eta - d\sigma_\tau)[y'(x) - y'(\xi)] + d\tau_{\tau\eta} \end{cases}$$

and thus, according to (1), since $\tau = x - \xi$ and $\eta = y(\xi) - y(x) - y'(\xi)(\xi - x)$:

$$\begin{cases} dT_n = -D_I(\xi)(\xi - x)^{-1}d\xi + D_{II}(\xi)[Y(x, \xi) - 2y'^*(\xi, x)]d\xi \\ dT_t = D_I(\xi)Y(x, \xi)d\xi - D_{II}(\xi)(\xi - x)^{-1}d\xi \end{cases} \quad (2)$$

where the following notations are used:

$$\begin{cases} y^*(x, \xi) = [y(x) - y(\xi)](x - \xi)^{-1} \\ y'^*(x, \xi) = [y'(x) - y'(\xi)](x - \xi)^{-1} \\ Y(x, \xi) = [y^*(x, \xi) - y'(\xi)](x - \xi)^{-1} \end{cases}$$

Now let $y(x)$ be an antisymmetric periodical function, describing an array of slightly curved, approximately collinear cuts along the x -axis. See Fig. 3.

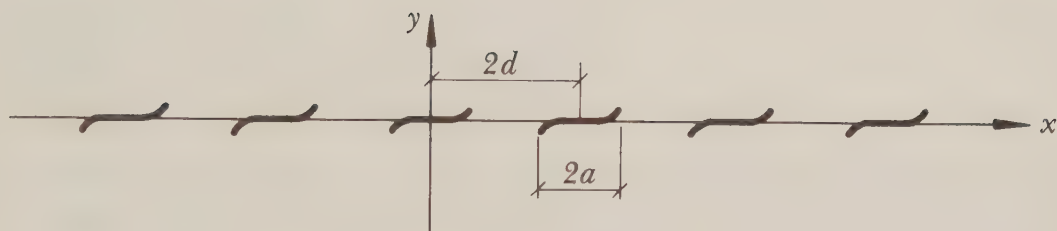


Fig. 3. A periodical array of slightly curved cracks.

$$y = y(x) = -y(-x) = y(x+2nd)$$

It is further assumed that the tractions on the crack surfaces obey the symmetry relations: $T_n(-x) = T_n(x)$, $T_t(-x) = T_t(x)$, and that T_n and T_t are periodical with period $2d$.

Let L be the union of the projection of all cuts on the x -axis:

$$L = \sum_{n=-\infty}^{\infty} L_n = \sum_{n=-\infty}^{\infty} \left\{ x: (2nd-a \leq x \leq 2nd+a) \right\}$$

The normal and tangential stresses at some point $(x, y(x))$ due to the total distribution of edge dislocations, consist of the sum of the contributions from all dislocations, i.e. an integral over L . Then, from (2):

$$\begin{cases} T_n(x) = -\oint_L (\xi-x)^{-1} D_I(\xi) d\xi + \oint_L D_{II}(\xi) [Y(x, \xi) - 2y'(x, \xi)] d\xi \\ T_t(x) = \oint_L D_I(\xi) Y(x, \xi) d\xi - \oint_L (\xi-x)^{-1} D_{II}(\xi) d\xi \end{cases} \quad (3)$$

where c on the integral sign denotes the Cauchy principal value. The condition that the cracks actually close at the ends is that

$$\int_{L_k} D_{Y,X}(x) dx = 0 \quad L_k = \left\{ x: (-a + 2kd) \leq x \leq (a + 2kd) \right\} \quad (4)$$

where

$$\begin{cases} D_Y(x) = D_I(x) + Y'(x) D_{II}(x) \\ D_X(x) = D_{II}(x) - Y'(x) D_I(x) \end{cases}$$

Since $|Y'(x)| \ll 1$, a first order solution to the integral equations with Cauchy kernels can be obtained by an iterative method. First a zero order solution to the problem, corresponding to $Y'(x) = 0$, is obtained, and this solution is then substituted back into (3).

First consider the zero order solution. By putting $Y'(x) = 0$ (3) reduces to

$$\begin{cases} T_n(x) = - \oint_L (\xi - x)^{-1} D_I^{(0)}(\xi) d\xi \\ T_t(x) = - \oint_L (\xi - x)^{-1} D_{II}^{(0)}(\xi) d\xi \end{cases} \quad (5)$$

The solution is [6]:

$$\begin{cases} D_I^{(0)}(x) = - [\pi^2 \omega_+(x)]^{-1} \left\{ \oint_L (x - \xi)^{-1} \omega_+(\xi) T_n(\xi) d\xi + \pi^2 k_I \right\} \\ D_{II}^{(0)}(x) = - [\pi^2 \omega_+(x)]^{-1} \left\{ \oint_L (x - \xi)^{-1} \omega_+(\xi) T_t(\xi) d\xi + \pi^2 k_{II} \right\} \end{cases} \quad (6)$$

k_I and k_{II} are constants and $\omega(x)$ is such that:

$$i) \quad D_{I,II}(x+2nd) = D_{I,II}(x) \quad n = 0, \pm 1, \pm 2, \dots$$

$$ii) \quad \omega_+(x) = -\omega_-(x) \text{ in the complex plane.}$$

Here $\omega_{\pm}(x) = \lim_{y \rightarrow \pm 0} \omega(x+iy)$, i.e. the upper and lower

limiting values of $\omega(z)$ as the x -axis is approached from the half-planes $y > 0$ and $y < 0$ respectively.

iii) $[\omega(x)]^{-1}$ is singular, but not stronger than $x^{-\frac{1}{2}}$, at $x = 2nd \pm a$, $n = 0, \pm 1, \pm 2, \dots$

iv) $[\omega(x)]^{-1}$ has no other singularities.

The integration path is taken on the upper side of L .

Here $\omega(z)$ is taken to be

$$\omega(z) = [\sin^2(\pi a/2d) - \sin^2(\pi z/2d)]^{\frac{1}{2}}, \quad \omega_+(0) = \sin(\pi a/2d)$$

which satisfies the above mentioned requirements.

The constants k_I and k_{II} in (6) are determined by (4):

$$k_{I,II} \int_{L_k} [\omega_+(x)]^{-1} dx = \int_{L_k} [\omega_+(x)]^{-1} \oint_L (\xi-x)^{-1} \omega_+(\xi) T_{n,t}(\xi) d\xi dx \quad (7)$$

Since $T_{n,t}(-\xi) = T_{n,t}(\xi)$, one obtains

$$\begin{aligned} \int_{L_k} [\omega_+(x)]^{-1} \oint_L (\xi-x)^{-1} \omega_+(\xi) T_{n,t}(\xi) d\xi dx = \\ = (1/2) \int_{L_k} [\omega_+(x)]^{-1} \oint_L [(\xi-x)^{-1} - (\xi+x)^{-1}] \omega_+(\xi) T_{n,t}(\xi) d\xi dx \end{aligned}$$

By making the substitutions $x \rightarrow x - 2kd$, $\xi \rightarrow \xi - 2kd$, one obtains

$$\int_{L_0} x [\omega_+(x)]^{-1} \oint_L (\xi^2 - x^2)^{-1} \omega_+(\xi) T_{n,t}(\xi) d\xi dx = 0$$

since the integrand is odd in x and L_0 is the symmetrical interval $-a \leq x \leq a$.

On the left side of (7):

$$\int_{L_k} [\omega_+(x)]^{-1} dx = 4d\pi^{-1} K[\sin(\pi a/2d)]$$

where $K(\alpha)$ is a complete elliptic integral of the first kind.

In Appendix 1 a list of integrals used is given. From (7) one obtains:

$$\begin{cases} k_I = 0 \\ k_{II} = 0 \end{cases}$$

Substitution of the zero order solution (6) back into (3) gives the first order approximation:

$$\begin{cases} \oint_L (\xi-x)^{-1} D_I(\xi) d\xi = -T_n(x) - \\ - \pi^2 \oint_L [\omega_+(\xi)]^{-1} \oint_L (\xi-\eta)^{-1} T_t(\eta) \omega_+(\eta) d\eta [Y(x, \xi) - 2y'^*(x, \xi)] d\xi \\ \oint_L (\xi-x)^{-1} D_{II}(\xi) d\xi = -T_t(x) - \pi^2 \oint_L [\omega_+(\xi)]^{-1} \oint_L (\xi-\eta)^{-1} T_n(\eta) \omega(\eta) d\eta Y(x, \xi) d\xi \end{cases}$$

with the solution

$$\begin{cases} \pi^2 \omega_+(x) D_I(x) = - \oint_L (x-\xi)^{-1} \omega_+(\xi) T_n(\xi) d\xi - \\ - \pi^2 \oint_L (x-\xi)^{-1} \omega_+(\xi) \oint_L [\omega_+(\eta)]^{-1} \oint_L (\eta-\tau)^{-1} \omega_+(\tau) T_t(\tau) d\tau [Y(\xi, \eta) - \\ - 2y'^*(\xi, \eta)] d\eta d\xi + C_I \\ \pi^2 \omega_+(x) D_{II}(x) = - \oint_L (x-\xi)^{-1} \omega_+(\xi) T_t(\xi) d\xi - \\ - \pi^2 \oint_L (x-\xi)^{-1} \omega_+(\xi) \oint_L [\omega_+(\eta)]^{-1} \oint_L (\eta-\tau)^{-1} \omega_+(\tau) T_t(\tau) d\tau Y(\xi, \eta) d\eta d\xi + \\ + C_{II} \end{cases} \quad (8)$$

Neglecting terms to the second or higher order of y and y' , and using the fact that

$$\int_{L_k} D_{I,II}^{(0)}(x) dx = 0$$

condition (4) can be written

$$\int_{L_k} [D_{I,II}(x) - D_{I,II}^{(0)}(x)] dx \pm \int_{L_k} Y'(x) D_{II,I}^{(0)}(x) dx = 0$$

where upper sign goes with first index.

This gives

$$\left\{ \begin{aligned} C_I \pi^{-2} \int_{L_k} [\omega_+(x)]^{-1} dx &= \\ &= \pi^{-4} \int_{L_k} [\omega_+(x)]^{-1} \oint_L (x-\xi)^{-1} \omega_+(\xi) \oint_L [\omega_+(\eta)]^{-1} \oint_L (\eta-\tau)^{-1} \omega_+(\tau) T_t(\tau) d\tau [Y(\xi, \eta) - \\ &- 2Y'(\xi, \eta)] d\eta d\xi dx + \pi^{-2} \int_{L_k} [\omega_+(x)]^{-1} Y'(x) \oint_L (x-\xi)^{-1} \omega_+(\xi) T_t(\xi) d\xi dx \\ C_{II} \pi^{-2} \int_{L_k} [\omega_+(x)]^{-1} dx &= \\ &= \pi^{-4} \int_{L_k} [\omega_+(x)]^{-1} \oint_L (x-\xi)^{-1} \omega_+(\xi) \oint_L [\omega_+(\eta)]^{-1} \oint_L (\eta-\tau)^{-1} \omega_+(\tau) T_n(\tau) d\tau \times \\ &\times Y(\xi, \eta) d\eta d\xi dx - \pi^{-2} \int_{L_k} [\omega_+(x)]^{-1} Y'(x) \oint_L (x-\xi)^{-1} \omega_+(\xi) T_t(\xi) d\xi dx \end{aligned} \right. \quad (9)$$

The variable changes $x \rightarrow x - 2kd$, $\xi \rightarrow \xi - 2kd$, $\eta \rightarrow \eta - 2kd$ and $\tau \rightarrow \tau - 2kd$ show that one can put $L_k = L_0$ in the expression for C_I and C_{II} , i.e. the integration over x is performed from $x = -a$ to $x = +a$.

Subsequent variable changes $\tau \rightarrow -\tau$, $\eta \rightarrow -\eta$ and $\xi \rightarrow -\xi$ show, that the integrands on right hand side in both equations (9) are odd in x , and thus the integrals vanish.

Since $\int_{L_k} [\omega_+(x)]^{-1} dx \neq 0$, this implies that

$$\left\{ \begin{aligned} C_I &= 0 \\ C_{II} &= 0 \end{aligned} \right.$$

Thus

$$\begin{aligned} \pi^2 \omega_+(x) D_{I, II}(x) &= - \oint_L (x-\xi)^{-1} \omega_+(\xi) T_{n,t}(\xi) d\xi - \\ &- \pi^{-2} \oint_L (x-\xi)^{-1} \omega_+(\xi) \oint_L [\omega_+(\eta)]^{-1} \oint_L (\eta-\tau)^{-1} \omega_+(\tau) T_{t,n}(\tau) d\tau [Y(\xi, \eta) - \\ &- 2\theta Y'^*(\xi, \eta)] d\eta d\xi \end{aligned}$$

where $\theta = 0$ in $D_{II}(x)$ and $\theta = 1$ in $D_I(x)$

Equation (10) can be simplified according to Appendix 2. The results reads

$$\begin{aligned} \pi^2 \omega_+(x) D_{I, II}(x) &= \oint_L (\xi-x)^{-1} \omega_+(\xi) T_{n,t}(\xi) d\xi + \\ &+ (1/2d) \oint_L [\omega_+(\xi) (\xi-x)]^{-1} Y(\xi) \cos(\pi\xi/2d) \oint_L (\xi-\eta)^{-1} \omega_+(\eta) T_{t,n}(\eta) d\eta d\xi - \\ &- (\pi^2/2d) Y(x) \cos(\pi x/2d) T_{t,n}(x) - \\ &- (1/\pi) \oint_L [\omega_+(\xi) (\xi-x)]^{-1} Y^*(\xi, x) [\sin(\pi\xi/2d) - \\ &- \sin(\pi x/2d)] \oint_L (\xi-\eta)^{-1} \omega_+(\eta) T_{t,n}(\eta) d\eta d\xi + \\ &+ (\pi/4d) \oint_L [\omega_+(\xi) (\xi-x)]^{-1} [Y(\xi) \sin(\pi\xi/d) - Y(a) \sin(\pi a/d)] T_{t,n}(\xi) d\xi - \\ &- \oint_L (\xi-x)^{-1} \omega_+(\xi) Y'(\xi) T_{t,n}(\xi) d\xi + \tag{11} \\ &+ \oint_L (\xi-x)^{-1} \omega_+(\xi) Y^*(\xi, x) T_{t,n}(\xi) d\xi - \\ &- (\pi/4d) [\omega_+(x)]^{-2} [Y(x) \sin(\pi x/d) - Y(a) \sin(\pi a/d)] \oint_L (\xi-x)^{-1} \omega_+(\xi) T_{t,n}(\xi) d\xi + \\ &+ 2\theta \oint_L \omega_+(\xi) Y'^*(\xi, x) T_{t,n}(\xi) d\xi + \\ &+ (1/\pi) (1-2\theta) \oint_L [\omega_+(\xi) (\xi-x)]^{-1} Y'(\xi) [\sin(\pi\xi/2d) - \\ &- \sin(\pi x/2d)] \oint_L (\xi-\eta)^{-1} \omega_+(\eta) T_{t,n}(\eta) d\eta d\xi \end{aligned}$$

Now assume that $T_{n,t}(x)$ to the first order in y' can be written

$$T_{n,t}(x) = T_{n,t}^{(0)} + T_{n,t}^{(1)} y'(x) \quad (12)$$

where $T_{n,t}^{(0)}$ and $T_{n,t}^{(1)}$ are constants and $y'(x)$ is even.

Substitution of (12) into (11) gives, neglecting terms of second or higher order in y and y' :

$$\begin{aligned} \pi^2 \omega_+(x) D_{I,II}(x) &= T_{n,t}^{(0)} \oint_L (\xi-x)^{-1} \omega_+(\xi) d\xi + \\ &+ T_{n,t}^{(1)} \oint_L (\xi-x)^{-1} \omega_+(\xi) y'(\xi) d\xi + \\ &+ (1/2d) T_{t,n}^{(0)} \oint_L [\omega_+(\xi) (\xi-x)]^{-1} y(\xi) \cos(\pi\xi/2d) \oint_L (\xi-\eta)^{-1} \omega_+(\eta) d\eta d\xi - \\ &- (\pi^2/2d) T_{t,n}^{(0)} y(x) \cos(\pi x/2d) - \\ &- (1/\pi) T_{t,n}^{(0)} \oint_L [\omega_+(\xi) (\xi-x)]^{-1} y'(\xi, x) [\sin(\pi\xi/2d) - \\ &- \sin(\pi x/2d)] \oint_L (\xi-\eta)^{-1} \omega_+(\eta) d\eta d\xi + \\ &+ (\pi/4d) T_{t,n}^{(0)} \oint_L [\omega_+(\xi) (\xi-x)]^{-1} [y(\xi) \sin(\pi\xi/d) - y(a) \sin(\pi a/d)] d\xi - \\ &- T_{t,n}^{(0)} \oint_L (\xi-x)^{-1} \omega_+(\xi) y'(\xi) d\xi + T_{t,n}^{(0)} \oint_L (\xi-x)^{-1} \omega_+(\xi) y^*(\xi, x) d\xi - \\ &- (\pi/4d) T_{t,n}^{(0)} [\omega_+(x)]^{-2} [y(x) \sin(\pi x/d) - y(a) \sin(\pi a/d)] \oint_L (\xi-x)^{-1} \omega_+(\xi) d\xi + \\ &+ 2\theta T_t^{(0)} \oint_L \omega_+(\xi) y'^*(\xi, x) d\xi + \\ &+ (1/\pi) (1-2\theta) T_{t,n}^{(0)} \oint_L [\omega_+(\xi) (\xi-x)]^{-1} y'(\xi) [\sin(\pi\xi/2d) - \\ &- \sin(\pi x/2d)] \oint_L (\xi-\eta)^{-1} \omega_+(\eta) d\eta d\xi \end{aligned} \quad (13)$$

After further simplifications of (13) according to Appendices 1 and 2, the final expressions for $D_I(x)$ and $D_{II}(x)$ are arrived at:

$$\begin{aligned}
\pi^2 \omega_+(x) D_{I,II}(x) &= -\pi T_{n,t}^{(0)} \sin(\pi x/2d) + \\
&+ (\pi/2d) [T_{n,t}^{(1)} + 2\theta T_t^{(0)}] \oint_{-a}^a \operatorname{cosec}[\pi(\xi-a)/2d] \omega_+(\xi) y'(\xi) d\xi + \\
&+ (\pi^2/2d) T_{t,n}^{(0)} y(x) \cos(\pi x/2d) + \\
&+ (\pi^2/4d) T_{t,n}^{(0)} [\omega_+(x)]^{-2} [y(x) \sin(\pi x/d) - y(a) \sin(\pi a/d)] \sin(\pi x/2d) + \\
&+ 2\pi\theta T_t^{(0)} y'(x) \sin(\pi x/2d)
\end{aligned} \tag{14}$$

Now the stress intensity factors at the crack tip $x=a$ are given by the limiting values

$$\begin{aligned}
K_{I,II}(a) &= \pi(2\pi)^{\frac{1}{2}} \lim_{x \rightarrow a} [(a-x)^{\frac{1}{2}} D_{I,II}(x)] = \\
&= -T_{t,n}^{(0)} [d \operatorname{cosec}(\pi a/d)]^{\frac{1}{2}} \sin(\pi a/2d) \left[(1-4\theta) y'(a) + \right. \\
&+ 2(T_{n,t}^{(0)}/T_{t,n}^{(0)}) - (\pi/d) y(a) \operatorname{cosec}(\pi a/d) + \\
&+ (1/d) (T_{n,t}^{(1)} + 2\theta T_t^{(0)}) / (T_{t,n}^{(0)}) \int_{-a}^a [\omega_+(\xi)]^{-1} y'(\xi) \cos(\pi \xi/2d) d\xi \left. \right]
\end{aligned} \tag{15}$$

These expressions are derived in Appendix 3.

Specialization to stress free crack surfaces and remote loading

Consider now the case of remote stresses σ_x^∞ , σ_y^∞ and τ_{xy}^∞ and stress free crack surfaces. By superposition of a homogeneous stress state $\sigma_x = -\sigma_x^\infty$, $\sigma_y = -\sigma_y^\infty$ and $\tau_{xy} = -\tau_{xy}^\infty$, the previously treated case is obtained, i.e. with zero remote stresses but tractions $T_n(x)$ and $T_t(x)$ on the crack surfaces. In fact, for $|y'(x)| \ll 1$:

$$\begin{aligned}
T_n(x) &= -\sigma_y^\infty + 2\tau_{xy}^\infty y'(x) \\
T_t(x) &= -\tau_{xy}^\infty - (\sigma_y^\infty - \sigma_x^\infty) y'(x)
\end{aligned}$$

A comparison with (12) then gives

$$\begin{aligned}
 T_n^{(0)} &= -\sigma_y^\infty \\
 T_n^{(1)} &= 2\tau_{xy}^\infty \\
 T_t^{(0)} &= -\tau_{xy}^\infty \\
 T_t^{(1)} &= -(\sigma_y^\infty - \sigma_x^\infty)
 \end{aligned} \tag{16}$$

Substituting (16) into (14), the expressions for $D_I(x)$ and $D_{II}(x)$ read:

$$\begin{aligned}
 \pi^2 \omega_+(x) D_I(x) &= \pi \sigma_y^\infty \sin(\pi x/2d) - (\pi^2/2d) \tau_{xy}^\infty y(x) \sin(\pi x/2d) - \\
 &- (\pi^2/4d) \tau_{xy}^\infty [\omega_+(x)]^{-2} [y(x) \sin(\pi x/d) - y(a) \sin(\pi a/d)] \sin(\pi x/2d) - \\
 &- 2\pi \tau_{xy}^\infty y'(x) \sin(\pi x/2d) \\
 \pi^2 \omega_+(x) D_{II}(x) &= \pi \tau_{xy}^\infty \sin(\pi x/2d) - \\
 &- (\pi/2d) (\sigma_y^\infty - \sigma_x^\infty) \int_{-a}^a \operatorname{cosec}[\pi(\xi-a)/2d] \omega_+(\xi) y'(\xi) d\xi - \\
 &- (\pi^2/2d) \sigma_y^\infty y(x) \cos(\pi x/2d) - \\
 &- (\pi^2/4d) \sigma_y^\infty [\omega_+(x)]^{-2} [y(x) \sin(\pi x/d) - y(a) \sin(\pi a/d)] \sin(\pi x/2d)
 \end{aligned}$$

The stress intensity factors for the crack tip at $x = a$ are given by substituting (16) into (15):

$$\begin{aligned}
 K_I(a) &= \tau_{xy}^\infty [d \operatorname{cosec}(\pi a/d)]^{\frac{1}{2}} \sin(\pi a/2d) \left[-3y'(a) + 2\sigma_y^\infty/\tau_{xy}^\infty - \right. \\
 &- \left. (\pi/d) y(a) \operatorname{cosec}(\pi a/d) \right] \\
 K_{II}(a) &= \sigma_y^\infty [d \operatorname{cosec}(\pi a/d)]^{\frac{1}{2}} \sin(\pi a/2d) \left[y'(a) + 2\tau_{xy}^\infty/\sigma_y^\infty - \right. \\
 &- \left. (\pi/d) y(a) \operatorname{cosec}(\pi a/d) + \right. \\
 &+ \left. (1/d) (1 - \sigma_x^\infty/\sigma_y^\infty) \int_{-a}^a [\omega(\xi)]^{-1} y'(\xi) \cos(\pi \xi/2d) d\xi \right]
 \end{aligned} \tag{17}$$

The case of one central crack $-a < x < a$, is arrived at by letting $a/d \ll 1$ in (17). Then

$$\begin{aligned} K_I(a) &= (\pi a)^{\frac{1}{2}} \sigma_Y^{\infty} - (3/2) (\pi a)^{\frac{1}{2}} \tau_{xy}^{\infty} y'(a) - (1/2) (\pi a)^{\frac{1}{2}} \tau_{xy}^{\infty} [y(a)/a] \\ K_{II}(a) &= (\pi a)^{\frac{1}{2}} \tau_{xy}^{\infty} + (1/2) (\pi a)^{\frac{1}{2}} \sigma_Y^{\infty} y'(a) - (1/2) (\pi a)^{\frac{1}{2}} \sigma_Y^{\infty} [y(a)/a] + \\ &+ (1/2d) (\pi a)^{\frac{1}{2}} (\sigma_Y^{\infty} - \sigma_X^{\infty}) \int_{-a}^a [\omega(\xi)]^{-1} y'(\xi) \cos(\pi \xi / 2d) d\xi \end{aligned} \quad (18)$$

These expressions agree with the results obtained by Gol'dstein and Salganik [2]. However, it should be observed that (18) (as well as (17)) is valid under the restriction $y(-x) = -y(x)$.

Stability under Mode I conditions of a straight crack path

The condition for growth is $K_{II}(a) = 0$ [7]. This gives, for the case of one single crack, according to (18):

$$y'(a) = y(a)/a - \lambda \int_0^a (a^2 - \xi^2)^{-\frac{1}{2}} y'(\xi) d\xi - 2\tau_{xy}^{\infty} / \sigma_Y^{\infty} \quad (19)$$

where

$$\lambda = (4/\pi) (1 - \sigma_X^{\infty} / \sigma_Y^{\infty})$$

The question of stability for the straight crack path needs to be clarified by a suitable definition of directional stability. One definition possible is to describe the straight path as stable if a slight deviation (due to some disturbance) at some stage of the crack growth remains local. This means that the crack tip at continued crack growth moves back towards the straight line in the original crack direction or, at least, that it does not move further away from this line. This definition, however, is rather pointless since it allows arbitrarily small deviations from the original direction to express lack of stability. Furthermore, especially in the case of a single crack, the deviation should be related to the length of the crack. An example is illustrating.

For the case $\tau_{xy}^{\infty}/\sigma_y^{\infty} = 0$ and $\sigma_x^{\infty}/\sigma_y^{\infty} = 1$ eq. (19) reads

$$y'(a) = y(a)/a \quad (20)$$

Assume now that there is a local deviation from the straight path such that

$$y(a) = y_0 \text{ for } a = a_0$$

Then, assuming no further disturbances, (20) predicts

$$y(a)/a = y_0/a_0 \text{ for } a \geq a_0$$

Thus, if the disturbance is small enough to be negligible, then the continued deviation (measured as $y(a)/a$) will also be negligible. However, the crack tip continues to move away from the (original) straight path, i.e. the definition discussed would express lack of stability. In fact, this would be the case also for lower values of $\sigma_x^{\infty}/\sigma_y^{\infty}$, the limiting value being $(1-\pi/4)$ at which (19) is satisfied by $y'(a) = 0$ for $a/a_0 \rightarrow \infty$, assuming a disturbance at $a \approx a_0$.

A more suitable definition of directional instability would be to demand that $[y(a)/a]/[y_0/a_0]$ should exceed any predetermined value if a/a_0 is sufficiently large and the initial deviation y_0/a_0 is infinitesimal. For the case of a single crack this means that stability prevails for $\sigma_x^{\infty}/\sigma_y^{\infty} \leq 1$ but not for $\sigma_x^{\infty}/\sigma_y^{\infty} > 1$. For the latter case, after putting

$$y(a) = aZ(a)$$

in Eq. (19), one obtains

$$\begin{aligned} aZ'(a) &= -\lambda \int_0^a (a^2 - \xi^2)^{-\frac{1}{2}} [\xi Z'(\xi) + Z(\xi)] d\xi = \\ &= -(\pi\lambda/2)Z(a) - \lambda \int_0^a Z'(\xi) [\xi(a^2 - \xi^2)^{-\frac{1}{2}} - \sin^{-1}(\xi/a)] d\xi \end{aligned} \quad (21)$$

where

$$\lambda = (4/\pi) (1 - \sigma_x^\infty / \sigma_y^\infty) < 0$$

The integral (and its integrand) in the right member of (21) is non-negative if $Z'(x)$ is non-negative. Thus, assuming a disturbance such that $y = y_0$ for $a = a_0$ one obtains

$$Z(a) \geq U(a)$$

if

$$aU'(a) = -(\pi\lambda/2)U(a) \text{ and } U(a_0) = y_0/a_0$$

This equation is satisfied by

$$U(a) = (y_0/a_0) (a/a_0)^{-\pi\lambda/2}$$

and thus

$$y(a) \geq y_0 (a/a_0)^{-\pi\lambda/2+1} = y_0 (a/a_0)^{2\sigma_x^\infty/\sigma_y^\infty-1}$$

The result shows that $[y(a)/a]/[y_0/a_0]$ will exceed any pre-determined value if a/a_0 is sufficiently large whenever $\sigma_x^\infty/\sigma_y^\infty > 1$. Hence, the straight path is unstable.

The approach is somewhat different from the one used by Cotterell and Rice [4] for a single crack. With another disturbance model than the present one they arrive at the result that the crack tip always continues to move away from the straight path, but since their solution formally shows that the tangent angle of the crack tends towards zero (i.e. parallelity with the original straight direction) under certain conditions, they conclude that the straight path then is stable. However, their solution is valid only up to a crack growth for which the tangent angle is still of the same order as the one originally imposed by the disturbance. Thus con-

sidering as an example a crack $-2a < x < 0$, $y = 0$ in a large plate subjected to the stresses $\sigma_x = \sigma_x^\infty$, $\sigma_y = \sigma_y^\infty$ one finds for positive x near the tip $x = 0$:

$$\begin{aligned} \sigma_x(x, 0) = & K_I (2\pi x)^{-\frac{1}{2}} + \sigma_x^\infty - \sigma_y^\infty + 3(\sigma_y^\infty)^2 (\pi x/2)^{\frac{1}{2}} (4K_I)^{-1} + \\ & + O(\sigma_y^\infty x^{\frac{1}{2}}/K_I)^{3/2} \end{aligned}$$

The right member can be truncated after the first two terms only if

$$(\sigma_y^\infty)^2 x^{\frac{1}{2}} (K_I |\sigma_x^\infty - \sigma_y^\infty|)^{-1} \ll 1$$

For $|\sigma_x^\infty - \sigma_y^\infty| \leq \sigma_y^\infty$ this requires that

$$|\sigma_x^\infty - \sigma_y^\infty| x^{\frac{1}{2}} (K_I)^{-1} \ll 1$$

or in the notation used by Cotterell and Rice [4]:

$$(T/k_I) x^{\frac{1}{2}} \ll 1$$

This is a necessary condition for neglect of the term $O(x^{\frac{1}{2}})$ in their Eq. (39). Then, however, the validity of the solution for $y(x) - \lambda(x)$ in their notation - is assured only for $x^{\frac{1}{2}} \ll k_I/T$. As can be seen from their Eq. (45), which (after correction of a misprint) reads

$$\lambda(x) = \theta_0 x \left[1 + \frac{8}{3} \frac{T}{K_I} \left(\frac{2x}{\pi} \right)^{\frac{1}{2}} + 4 \frac{T^2 x}{K_I^2} + \dots \right]$$

the validity is assured only as long as $\lambda'(x)$ remains of the same order as the initial angle θ_0 . Extension to $(T/k_I) x^{\frac{1}{2}} \gg 1$, which would be necessary for deciding whether or not $\lambda'(x)$ tends towards zero as the crack extends, is thus not possible. The conclusion in [4] about path stability is therefore conjectural.

The problem of path stability for a single crack was also studied

by Karihaloo, Keer, Nemat-Nasser and Oranratnachai [8], who used the perturbation method suggested by Cotterell and Rice [4]. They extended formally the validity of the solution by including one more term in the expression for K_{II} , but such an extension is not sufficient for determination of crack path stability.

The situation is different for a periodic array of cracks. The condition $K_{II}(a) = 0$ gives, from Eq. (17):

$$y'(a) = (\pi/d)y(a)\operatorname{cosec}(\pi a/d) - (2/d)(1-\sigma_x^\infty/\sigma_y^\infty) \int_0^a [\omega(\xi)]^{-1} y'(\xi) \cos(\pi \xi/2d) d\xi - 2\tau_{xy}^\infty/\sigma_y^\infty \quad (22)$$

For $\tau_{xy}^\infty/\sigma_y^\infty = 0$ and $\sigma_x^\infty/\sigma_y^\infty = 1$ Eq. (22) reads

$$y'(a) = (\pi/d)y(a)\operatorname{cosec}(\pi a/d) \quad (23)$$

Hence, assuming a disturbance such that $y(a) = y_0$ for $a = a_0$ one obtains

$$y(a) = y_0 \tan(\pi a/2d) / \tan(\pi a_0/2d)$$

Thus $[y(a)/a]/[y_0/a_0]$ exceeds any predetermined value as a/d approaches unity. Thus the straight crack path is unstable and tip to tip coalescence will not take place. It is immediately obvious that this situation also prevails for $\sigma_x^\infty/\sigma_y^\infty > 1$ since this implies a term added to the right member of (23) that is of the same sign as the first term.

As for the remaining case, $\sigma_x^\infty/\sigma_y^\infty < 1$, it is illuminating to write (22) in the form

$$(3 - 2\sigma_x^\infty/\sigma_y^\infty)y'(a) = (\pi/d)y(a)\operatorname{cosec}(\pi a/d) + (2/d)(1-\sigma_x^\infty/\sigma_y^\infty) \int_0^a [\omega(\xi)]^{-1} [y'(a) - y'(\xi)] \cos(\pi \xi/2d) d\xi$$

Assume now a deviation from the straight path such that

$$y'(a) = y'_0 = \text{constant for } a_0 - \delta < x < a_0 \text{ and } a_0 > d/2$$

Then

$$\begin{aligned} y'(a_0 + 0) &= (\pi y'_0 \delta / d) (3 - 2\sigma_x^\infty / \sigma_y^\infty)^{-1} \operatorname{cosec}(\pi a_0 / d) + \\ &+ (2/d) (1 - \sigma_x^\infty / \sigma_y^\infty) (3 - 2\sigma_x^\infty / \sigma_y^\infty)^{-1} \int_0^{a_0} [\omega(\xi)]^{-1} [y'_0(a_0 + 0) - \\ &- y'(\xi)] \cos(\pi \xi / 2d) d\xi \end{aligned} \quad (24)$$

Obviously $y'(a_0 + 0) > y'_0$ if

$$\sin(\pi a_0 / d) \leq (\pi \delta / d) (3 - 2\sigma_x^\infty / \sigma_y^\infty)^{-1}$$

At continued crack growth both terms of the right member of (24) are of the same sign and $[y(a)/a]/[y_0/a_0]$ increases beyond any predetermined value as a/d approaches unity. Hence even this case is unstable. i.e. regardless of the value of $\sigma_x^\infty / \sigma_y^\infty$ tip to tip coalescence will not occur. The cracks avoid each other.

In this paper complete antisymmetry, $y(-x) = -y(x)$, has been assumed, i.e. also for disturbances. However, it would be more logical to assume that a disturbance turns up at one end, only. Then, for the case of a single crack, $-a_0 < x < a_0$, the more general result by Gol'dstein and Salganik [2] can be used to show that a disturbance $y'(x) = y'_0$ for $-a_0 < x < -a_0 + \delta$ ($\delta/a_0 \ll 1$) at further crack growth gives rise to a deviation $y'(x) = y'_0 \delta / 2a_0 + y'_0 O(\delta/a_0)^{3/2}$ at the other end, $x = +a_0$, indicating that the crack path tends towards antisymmetry. This result together with the outcome of numerous experiments (e.g. those referred to in the caption to Fig. 1), should justify the assumption of antisymmetry.

It should be noticed that a local criterium (i.e. valid at the crack tip vicinity) is not appropriate: the only significant difference between different cases would then be the sign of σ_x at the crack surfaces. However, in the present case $\sigma_x = \sigma_x^\infty - \sigma_y^\infty$ along the crack surfaces, i.e. σ_x is independent of a_0/d at fixed σ_x^∞ and σ_y^∞ but path stability nevertheless depends upon a_0/d . On the other hand the influence from other cracks on the stress fields at a crack tip ought to be dominated by the nearest one. Thus it can be assumed that the essential results can be carried over to non-periodic arrays, for instance to the case of two originally collinear cracks.

SUMMARY

It is found that instability of the straight crack paths will occur in a periodic array of originally collinear, slowly growing cracks. Thus tip to tip coalescence will not take place. It can be assumed that the result holds also for non-periodic arrays, for instance for two originally collinear cracks.

The result might be of some interest in discussions on coalescence of micro-cracks in front of a main crack.

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APPENDIX 1

Integrals

In this appendix a list of integrals used will be given. The technique used at calculating these integrals is shown by one example.

List of integrals:

$$\oint_L (\xi - x)^{-1} \omega_+(\xi) d\xi = -\pi \sin(\pi x/2d)$$

$$\oint_L [(\xi - x)(\xi - \tau)]^{-1} \omega_+(\xi) d\xi = -\pi (x - \tau)^{-1} [\sin(\pi x/2d) - \sin(\pi \tau/2d)]$$

$$\oint_L [(\xi - x) \omega_+(\xi)]^{-1} d\xi = 0$$

$$\oint_L [(\xi - x)(\xi - \tau) \omega_+(\xi)]^{-1} d\xi = 0$$

$$\oint_L [(\xi - x) \omega_+(\xi)]^{-1} \cos(\pi \xi/2d) d\xi = 0$$

$$\oint_L [(\xi - x) \omega_+(\xi)]^{-1} \sin(\pi \xi/2d) d\xi = 0$$

$$\oint_L [(\xi - x)(\xi - \tau) \omega_+(\xi)]^{-1} \sin(\pi \xi/2d) d\xi = 0$$

$$\oint_L [(\xi - x) \omega_+(\xi)]^{-1} \sin(\pi \xi/d) d\xi = 2\pi \cos(\pi x/2d)$$

$$\int_{L_k} [\omega_+(\xi)]^{-1} d\xi = 4d\pi^{-1} K[\sin(\pi a/2d)]$$

$$\int_{L_k} [\omega_+(\xi)]^{-1} \sin(\pi \xi/2d) d\xi = 0$$

$$\int_L [\omega_+(\xi)]^{-1} \sin(\pi \xi/d) d\xi = 0$$

where
$$\begin{cases} L_k = \{x: 2kd - a \leq x \leq 2kd + a\} \\ L = \{x: \sum_{n=-\infty}^{\infty} (2nd - a \leq x \leq 2nd + a)\} \\ K(\alpha) \text{ is the complete elliptical integral of the first kind.} \end{cases}$$

Example

Consider $J = \oint_L (\xi - x)^{-1} \omega_+(\xi) d\xi$ $x \in L = \left\{ x: \sum_{n=-\infty}^{\infty} (2nd - a < x < 2nd + a) \right\}$

The integration path chosen in the complex plane is shown in Fig. 6, where $L' = \sum_{i=1}^6 L_i$ and $\rho \gg d$.

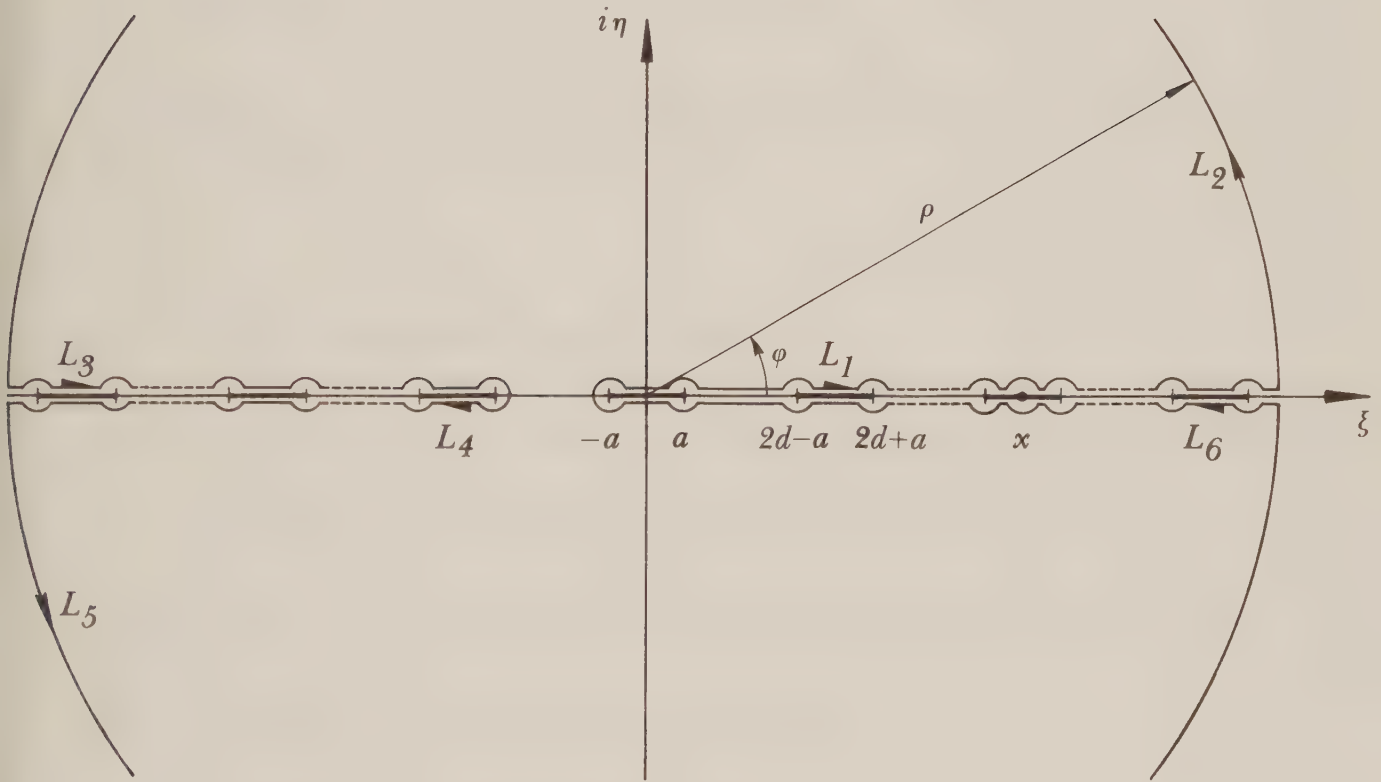


Fig. 6. Path of integration in the ζ -plane.

Put $\omega(\zeta) = [\sin^2(\pi a/2d) - \sin^2(\pi \zeta/2d)]^{\frac{1}{2}}$. Then $\omega_+(\xi) = -\omega_-(\xi)$, where $\omega_+(\xi)$ and $\omega_-(\xi)$ denote the limiting values of $\omega(\zeta)$ as the ξ -axis is approached from $\eta > 0$ and $\eta < 0$ respectively.

Now put $L'' = L_1 + L_3 + L_4 + L_6$

Then

$$2J \rightarrow \oint_{L''} (\xi-x)^{-1} \omega_+(\xi) d\xi \text{ as } \rho/d \rightarrow \infty$$

since the contributions of the integrals from the parts between the cuts $2nd-a \leq x \leq 2nd+a$ cancel each other.

Thus

$$2J + \int_{L_2} (\zeta-x)^{-1} \omega_+(\zeta) d\zeta + \int_{L_5} (\zeta-x)^{-1} \omega_+(\zeta) d\zeta \rightarrow 0 \text{ as } \rho/d \rightarrow \infty \quad (A1:1)$$

according to the Cauchy integral theorem, since no singularities of the integrand are enclosed by L' .

Now

$$\begin{aligned} \omega(\zeta) &= [\sin^2(\pi a/2d) - \sin^2(\pi \zeta/2d)]^{\frac{1}{2}} = \\ &= \left[\sin^2(\pi a/2d) - (2i)^{-2} [\exp(i\pi \zeta/2d) - \exp(-i\pi \zeta/2d)]^2 \right]^{\frac{1}{2}} = \\ &= (1/2) [R+iI]^{\frac{1}{2}} = (1/2) [R^2+I^2]^{1/4} \exp[(i/2) \tan^{-1}(I/R)] \end{aligned}$$

where R and I are real. For $\zeta = \rho \exp(i\phi)$ one obtains

$$\begin{aligned} R &= 4\sin^2(\pi a/2d) + 2 + [\exp(\pi \rho d^{-1} \sin \phi) + \\ &+ \exp(-\pi \rho d^{-1} \sin \phi)] \cos(\pi \rho d^{-1} \cos \phi) \end{aligned}$$

$$I = -[\exp(\pi \rho d^{-1} \sin \phi) - \exp(-\pi \rho d^{-1} \sin \phi)] \sin(\pi \rho d^{-1} \cos \phi)$$

For $0 < \phi < \pi$:

$$R \rightarrow \exp(\pi \rho d^{-1} \sin \phi) \cos(\pi \rho d^{-1} \cos \phi) \text{ as } \rho/d \rightarrow \infty$$

$$I \rightarrow -\exp(\pi \rho d^{-1} \sin \phi) \sin(\pi \rho d^{-1} \cos \phi) \text{ as } \rho/d \rightarrow \infty$$

Then

$$[R^2 + I^2]^{-\frac{1}{4}} \rightarrow \exp[\pi \rho (2d)^{-1} \sin \phi] \text{ as } \rho/d \rightarrow \infty$$

and

$$\begin{aligned} \exp[(i/2) \tan^{-1}(I/R)] &\rightarrow \exp\left[-(i/2) \tan^{-1}[\tan(\pi \rho d^{-1} \cos \phi)]\right] = \\ &= \exp[-i \pi \rho (2d)^{-1} \cos \phi] \text{ as } \rho/d \rightarrow \infty \end{aligned}$$

Thus

$$\omega(\zeta) \rightarrow (1/2) \exp[\pi \rho (2d)^{-1} (\sin \phi - i \cos \phi)] \text{ as } \rho/d \rightarrow \infty, \quad 0 < \phi < \pi \quad (\text{A1:2})$$

Similar treatment of $\omega(\zeta)$ in the case $\pi < \phi < 2\pi$ gives

$$\omega(\zeta) \rightarrow -(1/2) \exp[-\pi \rho (2d)^{-1} (\sin \phi - i \cos \phi)] \text{ as } \rho/d \rightarrow \infty, \quad \pi < \phi < 2\pi \quad (\text{A1:3})$$

Return to (A1:1). Using (A1:2) and (A1:3) one gets

$$\begin{aligned} J &= -(1/2) \left[\int_{L_2}^{\pi} (\zeta - x)^{-1} \omega(\zeta) d\zeta + \int_{L_5}^{\pi} (\zeta - x)^{-1} \omega(\zeta) d\zeta \right] = \\ &= -(i/2) \int_0^{\pi} [\rho \exp(i\phi) - x]^{-1} \exp[\pi \rho (2d)^{-1} (\sin \phi - i \cos \phi)] \rho \exp(i\phi) d\phi + \\ &+ (i/2) \int_{\pi}^{2\pi} [\rho \exp(i\phi) - x]^{-1} \exp[-\pi \rho (2d)^{-1} (\sin \phi - i \cos \phi)] \rho \exp(i\phi) d\phi \end{aligned}$$

Change variable of integration in the last integral,

$\phi \rightarrow \theta = \phi - \pi$, and add the integrals:

$$\begin{aligned} J &= -(i/2) \int_0^{\pi} \exp[\pi \rho (2d)^{-1} (\sin \phi - i \cos \phi)] [(x/\rho) \exp(-i\phi)] [1 - \\ &- (x/\rho)^2 \exp(-2i\phi)] d\phi = \\ &= -(i/2) \sum_{n=0}^{\infty} \left\{ (x/\rho)^{2n+1} \int_0^{\pi} \exp[\pi \rho (2d)^{-1} \sin \phi] \cos[\pi \rho (2d)^{-1} \cos \phi + \right. \\ &+ (2n+1)\phi] d\phi \left. \right\} - \\ &- (1/2) \sum_{n=0}^{\infty} \left\{ (x/\rho)^{2n+1} \int_0^{\pi} \exp[\pi \rho (2d)^{-1} \sin \phi] \sin[\pi \rho (2d)^{-1} \cos \phi + \right. \\ &+ (2n+1)\phi] d\phi \left. \right\} \end{aligned}$$

These two integrals can be evaluated [9]. The first one is zero and the second one equals

$$\begin{aligned} & \int_0^\pi \exp [\pi \rho (2d)^{-1} \sin \phi] \sin [\pi \rho (2d)^{-1} \cos \phi + (2n+1) \phi] d\phi = \\ & = 2\pi [\pi \rho (2d)^{-1}]^{2n+1} (-1)^n [(2n+1)!]^{-1} \end{aligned}$$

Thus

$$J = -\pi \sum_{n=0}^{\infty} (\pi x/2d) (-1)^n [(2n+1)!]^{-1} = -\pi \sin(\pi x/2d)$$

APPENDIX 2

Simplification of the expressions for $D_I(x)$ and $D_{II}(x)$

$D_I(x)$ and $D_{II}(x)$ are given by

$$\begin{aligned} \pi^2 \omega(x) D_{I,II}(x) = & - \oint_L (x-\xi)^{-1} \omega_+(\xi) T_{n,t}(\xi) d\xi - \\ & - \pi^{-2} \oint_L (x-\xi)^{-1} \omega_+(\xi) \oint_L [\omega_+(\eta)]^{-1} \oint_L (\eta-\tau)^{-1} \omega_+(\tau) T_{t,n}(\tau) [Y(\xi, \eta) - \\ & - 2\theta y'^*(\xi, \eta)] d\eta d\xi \end{aligned}$$

where

$$\left\{ \begin{aligned} \theta &= 0 \text{ in } D_{II}(x) \text{ and } \theta = 1 \text{ in } D_I(x) \\ Y(x, \xi) &= (x-\xi)^{-1} [y^*(x, \xi) - y'(\xi)] \\ y^*(x, \xi) &= (x-\xi)^{-1} [y(x) - y(\xi)] \\ y'^*(x, \xi) &= (x-\xi)^{-1} [y'(x) - y'(\xi)] \\ \omega_+(\xi) &= [\sin^2(\pi a/2d) - \sin^2(\pi \xi/2d)]^{\frac{1}{2}} \\ L &= \sum_{n=-\infty}^{\infty} L_k = \sum_{n=-\infty}^{\infty} \{x: (2nd-a \leq x \leq 2nd+a)\} \end{aligned} \right.$$

In the following, the notation $\omega(x)$ will be used to mean $\omega_+(x)$.

Consider

$$\begin{aligned} I = & \oint_L (x-\xi)^{-1} \omega(\xi) \oint_L [\omega(\eta)]^{-1} \oint_L (\eta-\tau)^{-1} \omega(\tau) T(\tau) [Y(\xi, \eta) - \\ & - 2\theta y'^*(\xi, \eta)] d\eta d\xi \end{aligned}$$

Change the order of integration [10] and split I into two parts:

$$\begin{aligned} I = & \oint_L \omega(\tau) T(\tau) \oint_L [\omega(\eta) (\eta-\tau)]^{-1} \oint_L (x-\xi)^{-1} \omega(\xi) Y(\xi, \eta) d\xi d\eta d\tau - \\ & - 2\theta \oint_L \omega(\tau) T(\tau) \oint_L [\omega(\eta) (\eta-\tau)]^{-1} \oint_L (x-\xi)^{-1} \omega(\xi) y'^*(\xi, \eta) d\xi d\eta d\tau = \end{aligned}$$

$$= \oint_L \omega(\tau) T(\tau) I_0 d\tau - 2\theta \oint_L \omega(\tau) T(\tau) J_0 d\tau \quad (A2:1)$$

J_0 will be considered first.

$$\begin{aligned} J_0 &= \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (x - \xi)^{-1} \omega(\xi) y'^*(\xi, \eta) d\xi d\eta = \\ &= -\oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (\xi - \eta)^{-1} y'^*(x, \xi) \omega(\xi) d\xi d\eta + \\ &+ y'(x) \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (\xi - \eta) (x - \xi)^{-1} \omega(\xi) d\xi d\eta - \\ &- \oint_L [\omega(\eta) (\eta - \tau)]^{-1} y'(\eta) \oint_L (\xi - \eta) (x - \xi)^{-1} \omega(\xi) d\xi d\eta \end{aligned}$$

Now change order of integration in the first term. The second term and the inner integral of the third term can be evaluated according to Appendix 1.

Thus

$$\begin{aligned} J_0 &= -\oint_L y'^*(x, \xi) \omega(\xi) \oint_L [\omega(\eta) (\eta - \tau) (\xi - \eta)]^{-1} d\eta d\xi + \pi^2 y'^*(x, \tau) + \\ &+ \pi y'(x) \oint_L [\omega(\eta) (\eta - \tau) (x - \eta)]^{-1} [\sin(\pi x/2d) - \sin(\pi \eta/2d)] d\eta - \\ &- \pi \oint_L [\omega(\eta) (\eta - \tau) (x - \eta)]^{-1} [\sin(\pi x/2d) - \sin(\pi \eta/2d)] y'(\eta) d\eta = \\ &= \pi^2 y'^*(x, \tau) + \pi \oint_L [\omega(\eta) (\eta - \tau)]^{-1} y'^*(x, \eta) [\sin(\pi x/2d) - \sin(\pi \eta/2d)] d\eta \end{aligned} \quad (A2:2)$$

since

$$\oint_L [\omega(\eta) (\eta - \tau) (\xi - \eta)]^{-1} d\eta = 0 \text{ according to Appendix 1.}$$

From (A2:1) one gets

$$\begin{aligned} I_0 &= \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (x - \xi)^{-1} \omega(\xi) Y(\xi, \eta) d\xi d\eta = \\ &= -\oint_L [\omega(\eta) (\eta - \tau)]^{-1} y'(\eta) \oint_L (x - \xi) (\xi - \eta)^{-1} \omega(\xi) d\xi d\eta - \\ &- \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (x - \xi) (\eta - \xi)^{-1} \omega(\xi) y^*(\xi, \eta) d\xi d\eta = -I_1 - I_2 \end{aligned} \quad (A2:3)$$

I_1 can be simplified according to Appendix 1:

$$I_1 = \pi \oint_L [\omega(\eta) (\eta - \tau) (x - \eta)]^{-1} y'(\eta) [\sin(\pi x/2d) - \sin(\pi \eta/2d)] d\eta \quad (A2:4)$$

Now

$$\begin{aligned} [(\eta - \tau) (x - \xi) (\eta - \xi)]^{-1} &\equiv (x - \tau)^{-1} \left[[(\eta - \tau) (\eta - \xi)]^{-1} - [(\eta - \tau) (x - \xi)]^{-1} - \right. \\ &\left. - [(\eta - x) (\eta - \xi)]^{-1} + [(\eta - x) (x - \xi)]^{-1} \right] \end{aligned}$$

Then

$$\begin{aligned} I_2 &= (x - \tau)^{-1} \left[\oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (\eta - \xi)^{-1} \omega(\xi) y^*(\xi, \eta) d\xi d\eta - \right. \\ &- \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (x - \xi)^{-1} \omega(\xi) y^*(\xi, \eta) d\xi d\eta - \\ &- \oint_L [\omega(\eta) (\eta - x)]^{-1} \oint_L (\eta - \xi)^{-1} \omega(\xi) y^*(\xi, \eta) d\xi d\eta + \\ &\left. + \oint_L [\omega(\eta) (\eta - x)]^{-1} \oint_L (x - \xi)^{-1} \omega(\xi) y^*(\xi, \eta) d\xi d\eta \right] = \\ &= (x - \tau)^{-1} [I_{11} - I_{12} - I_{13} + I_{14}] \end{aligned} \quad (A2:5)$$

First consider I_{11}

$$I_{11} = \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (\eta - \xi)^{-2} \omega(\xi) [y(\eta) - y(\xi)] d\xi d\eta$$

Integrate the inner integral by parts:

$$\begin{aligned} I_{11} &= \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \left[\left\{ (\eta - \xi)^{-1} \omega(\xi) [y(\eta) - y(\xi)] \right\} - \oint_L (\eta - \xi)^{-1} \omega'(\xi) [y(\eta) - \right. \\ &- y(\xi)] d\xi + \oint_L (\eta - \xi)^{-1} \omega(\xi) y'(\xi) d\xi \Big] d\eta = \\ &= - \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (\eta - \xi)^{-1} \omega'(\xi) [y(\eta) - y(\xi)] d\xi + \\ &+ \oint_L \omega(\xi) y'(\xi) \oint_L [\omega(\eta) (\eta - \tau) (\eta - \xi)]^{-1} d\eta d\xi + \pi^2 y'(\tau) = \end{aligned}$$

$$= - \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (\eta - \xi)^{-1} \omega'(\xi) [y(\eta) - y(\xi)] d\xi d\eta + \pi^2 y'(\tau)$$

since the expression inside brackets $\left\{ \right\}$ is zero at the end-points of L .

In order to make some changes of the order of integration possible both the inner and the outer integral of I_{11} are transformed to the interval L_0 .

The inner integral

$$\begin{aligned} \oint_L (\eta - \xi)^{-1} \omega'(\xi) [y(\eta) - y(\xi)] d\xi &= \sum_{-\infty}^{\infty} \oint_{-a+2nd}^{a+2nd} (\eta - \xi)^{-1} \omega'(\xi) [y(\eta) - y(\xi)] d\xi = \\ &= \oint_{-a}^a \omega'(\xi) [y(\eta) - y(\xi)] \left[\sum_{-\infty}^{\infty} (-1)^n (\eta - \xi - 2nd)^{-1} \right] d\xi = \\ &= (\pi/2d) \oint_{-a}^a \omega'(\xi) \operatorname{cosec}[\pi(\eta - \xi)/2d] [y(\eta) - y(\xi)] d\xi \end{aligned}$$

Similar treatment of the outer integral gives

$$\begin{aligned} \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_{-a}^a \omega'(\xi) \operatorname{cosec}[\pi(\eta - \xi)/2d] [y(\eta) - y(\xi)] d\xi d\eta &= \\ &= \oint_{-a}^a [\omega(\eta)]^{-1} \left[\sum_{-\infty}^{\infty} (\eta - \tau + 2nd)^{-1} \right] \oint_{-a}^a \omega'(\xi) \operatorname{cosec}[\pi(\eta - \xi)/2d] [y(\eta) - \\ &- y(\xi)] d\xi d\eta = \\ &= (\pi/2d) \oint_{-a}^a \left[\omega(\eta) \tan[\pi(\eta - \tau)/2d] \right]^{-1} \oint_{-a}^a \omega'(\xi) \operatorname{cosec}[\pi(\eta - \xi)/2d] [y(\eta) - \\ &- y(\xi)] d\xi d\eta \end{aligned}$$

which gives

$$\begin{aligned} I_{11} &= -(\pi/2d)^2 \oint_{-a}^a \left[\omega(\eta) \tan[\pi(\eta - \tau)/2d] \right]^{-1} \oint_{-a}^a \omega'(\xi) \operatorname{cosec}[\pi(\eta - \xi)/2d] [y(\eta) - \\ &- y(\xi)] d\xi d\eta + \pi^2 y'(\tau) \end{aligned}$$

Now write $\omega'(\xi)$ in the form

$$\begin{aligned}
\omega'(\xi) &= \frac{d}{d\xi} \left\{ [\sin^2(\pi a/2d) - \sin^2(\pi \xi/2d)]^{-\frac{1}{2}} \right\} = -\pi [4d\omega(\xi)]^{-1} \sin(\pi \xi/d) = \\
&= -\pi (4d)^{-1} \cos(\pi \xi/2d) \left[[\sin(\pi a/2d) + \sin(\pi \xi/2d)]^{\frac{1}{2}} [\sin(\pi a/2d) - \right. \\
&- \sin(\pi \xi/2d)]^{-\frac{1}{2}} - \\
&- [\sin(\pi a/2d) - \sin(\pi \xi/2d)]^{\frac{1}{2}} [\sin(\pi a/2d) + \sin(\pi \xi/2d)]^{-\frac{1}{2}} \left. \right] = \\
&= -\pi (4d)^{-1} \cos(\pi \xi/2d) [Q_1(\xi) - Q_2(\xi)]
\end{aligned}$$

With care taken to avoid divergent integrals, I_{11} can be decomposed into five parts:

$$\begin{aligned}
I_{11} &= (\pi^3/16d^3) \left\{ \oint_{-a}^a \left[\omega(\eta) \tan[\pi(\eta-\tau)/2d] \right]^{-1} [y(\eta) - \right. \\
&- y(a)] \oint_{-a}^a Q_1(\xi) \cos(\pi \xi/2d) \operatorname{cosec}[\pi(\eta-\xi)/2d] d\eta d\xi - \\
&- \oint_{-a}^a \left[\omega(\eta) \tan[\pi(\eta-\tau)/2d] \right]^{-1} \oint_{-a}^a Q_1(\xi) \cos(\pi \xi/2d) \operatorname{cosec}[\pi(\eta-\xi)/2d] [y(\xi) - \\
&- y(a)] d\xi d\eta - \\
&- \oint_{-a}^a \left[\omega(\eta) \tan[\pi(\eta-\tau)/2d] \right]^{-1} [y(\eta) - \\
&- y(-a)] \oint_{-a}^a Q_2(\xi) \cos(\pi \xi/2d) \operatorname{cosec}[\pi(\eta-\xi)/2d] d\xi d\eta + \\
&+ \int_{-a}^a \left[\omega(\eta) \tan[\pi(\eta-\tau)/2d] \right]^{-1} \oint_{-a}^a Q_2(\xi) \cos(\pi \xi/2d) \operatorname{cosec}[\pi(\eta-\xi)/2d] [y(\xi) - \\
&- y(-a)] d\xi d\eta \left. \right\} + \pi^2 y'(\tau)
\end{aligned}$$

Since $y(a) = -y(-a)$, one gets, by adding the first and the third terms, and changing order of integration in the rest:

$$\begin{aligned}
I_{11} &= (\pi^3/16d^3) \left\{ \oint_{-a}^a \left[\omega(\eta) \tan[\pi(\eta-\tau)/2d] \right]^{-1} y(\eta) \int_{-a}^a \left[\omega(\xi) \sin[\pi(\eta-\xi)/2d] \right]^{-1} \right. \\
&\times \sin(\pi \xi/d) d\xi d\eta -
\end{aligned}$$

$$= \oint_L \omega(\tau) T(\tau) I_0 d\tau - 2\theta \oint_L \omega(\tau) T(\tau) J_0 d\tau \quad (A2:1)$$

J_0 will be considered first.

$$\begin{aligned} J_0 &= \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (x - \xi)^{-1} \omega(\xi) y'^*(\xi, \eta) d\xi d\eta = \\ &= -\oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (\xi - \eta)^{-1} y'^*(x, \xi) \omega(\xi) d\xi d\eta + \\ &+ y'(x) \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L [(\xi - \eta) (x - \xi)]^{-1} \omega(\xi) d\xi d\eta - \\ &- \oint_L [\omega(\eta) (\eta - \tau)]^{-1} y'(\eta) \oint_L [(\xi - \eta) (x - \xi)]^{-1} \omega(\xi) d\xi d\eta \end{aligned}$$

Now change order of integration in the first term. The second term and the inner integral of the third term can be evaluated according to Appendix 1.

Thus .

$$\begin{aligned} J_0 &= -\oint_L y'^*(x, \xi) \omega(\xi) \oint_L [\omega(\eta) (\eta - \tau) (\xi - \eta)]^{-1} d\eta d\xi + \pi^2 y'^*(x, \tau) + \\ &+ \pi y'(x) \oint_L [\omega(\eta) (\eta - \tau) (x - \eta)]^{-1} [\sin(\pi x/2d) - \sin(\pi \eta/2d)] d\eta - \\ &- \pi \oint_L [\omega(\eta) (\eta - \tau) (x - \eta)]^{-1} [\sin(\pi x/2d) - \sin(\pi \eta/2d)] y'(\eta) d\eta = \\ &= \pi^2 y'^*(x, \tau) + \pi \oint_L [\omega(\eta) (\eta - \tau)]^{-1} y'^*(x, \eta) [\sin(\pi x/2d) - \sin(\pi \eta/2d)] d\eta \end{aligned} \quad (A2:2)$$

since

$\oint_L [\omega(\eta) (\eta - \tau) (\xi - \eta)]^{-1} d\eta = 0$ according to Appendix 1.

From (A2:1) one gets

$$\begin{aligned} I_0 &= \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L (x - \xi)^{-1} \omega(\xi) y(\xi, \eta) d\xi d\eta = \\ &= -\oint_L [\omega(\eta) (\eta - \tau)]^{-1} y'(\eta) \oint_L [(x - \xi) (\xi - \eta)]^{-1} \omega(\xi) d\xi d\eta - \\ &- \oint_L [\omega(\eta) (\eta - \tau)]^{-1} \oint_L [(x - \xi) (\eta - \xi)]^{-1} \omega(\xi) y^*(\xi, \eta) d\xi d\eta = -I_1 - I_2 \end{aligned} \quad (A2:3)$$

$$\begin{aligned}
& + \cos(\pi a/2d) \int_{-a}^a \left[\omega(\eta) \tan[\pi(\eta-\tau)/2d] \right]^{-1} \int_{-a}^a \left[\omega(\eta) \sin[\pi(\eta-\xi)/2d] \right]^{-1} \times \\
& \times d\xi d\eta \Big\} = \\
& = -2\sin(\pi a/2d) y(a) \left[J_{11} + J_{12} \right]
\end{aligned}$$

Change order of integration in J_{11} :

$$\begin{aligned}
J_{11} &= \int_{-a}^a [\omega(\xi)]^{-1} [\cos(\pi\xi/2d) - \\
& - \cos(\pi a/2d)] \int_{-a}^a \left[\omega(\eta) \tan[\pi(\eta-\tau)/2d] \sin[\pi(\eta-\tau)/2d] \right]^{-1} d\eta d\xi + \\
& + 4d^2 [\omega(\tau)]^{-2} [\cos(\pi\tau/2d) - \cos(\pi a/2d)] = \\
& = 4d^2 [\omega(\tau)]^2 [\cos(\pi\tau/2d) - \cos(\pi a/2d)] = \\
& = 4d^2 [\cos(\pi\tau/2d) - \cos(\pi a/2d)]^{-1}
\end{aligned}$$

since

$$\int_{-a}^a \left[\omega(\eta) \tan[\pi(\eta-\tau)/2d] \sin[\pi(\eta-\tau)/2d] \right]^{-1} d\eta = 0$$

which is easily seen by making the substitution $u = \sin\eta$.

The same substitution, $u = \sin\eta$, shows that

$$J_{12} = 0$$

The two remaining double integrals in the expression for I_{11} both equal zero and thus

$$\begin{aligned}
I_{11} &= -(\pi^2/2d) \int_L [\omega(\eta) (\eta-\tau)]^{-1} y(\eta) \cos(\pi\eta/2d) d\eta - \\
& - (\pi^3/4d) [\omega(\tau)]^{-2} [y(\tau) \sin(\pi\tau/d) - y(a) \sin(\pi a/d)] + \pi^2 y'(\tau)
\end{aligned}$$

I_{13} can be arrived at by exchanging τ by x in I_{11} .

Thus

$$\begin{aligned} I_{13} &= (I_{11})_{\tau \rightarrow x} = -(\pi^2/2d) \oint_L [\omega(\eta) (\eta-x)]^{-1} y(\eta) \cos(\pi\eta/2d) d\eta - \\ &- (\pi^3/4d) [\omega(x)]^{-2} [y(x) \sin(\pi x/d) - y(a) \sin(\pi a/d)] + \pi^2 y'(x) \end{aligned} \quad (A2:7)$$

I_{12} is treated similarly as I_{11} .

$$\begin{aligned} I_{12} &= \oint_L [\omega(\eta) (\eta-\tau)]^{-1} \oint_L (x-\xi)^{-1} \omega(\xi) y^*(\eta, \xi) d\xi d\eta = \\ &= \oint_L [\omega(\eta) (\eta-\tau)]^{-1} \oint_L [(x-\xi) (\eta-\xi)]^{-1} \omega(\xi) [y(\eta) - y(\xi)] d\xi d\eta = \\ &= \oint_L [\omega(\eta) (\eta-\tau)]^{-1} y(\eta) \oint_L [(x-\xi) (\eta-\xi)]^{-1} \omega(\xi) d\xi d\eta - \\ &- \oint_L [\omega(\eta) (\eta-\tau)]^{-1} \oint_L [\eta-\xi) (x-\xi)]^{-1} [y(\xi) - y(x)] \omega(\xi) d\xi d\eta - \\ &- y(x) \oint_L [\omega(\eta) (\eta-\tau)]^{-1} \oint_L [(\eta-\xi) (x-\xi)]^{-1} \omega(\xi) d\xi d\eta \end{aligned}$$

Change order of integration in the second term and evaluate integrals according to Appendix 1.

Then:

$$\begin{aligned} I_{12} &= -\pi \oint_L [\omega(\eta) (\eta-\tau) (\eta-x)]^{-1} y(\eta) [\sin(\pi\eta/2d) - \sin(\pi x/2d)] d\eta - \\ &- \oint_L (x-\xi)^{-1} [y(\xi) - y(x)] \omega(\xi) \oint_L [\omega(\eta) (\eta-\tau) (\eta-\xi)]^{-1} d\eta d\xi + \\ &+ \pi^2 y^*(\tau, x) + \pi y(x) \oint_L [\omega(\eta) (\eta-\tau) (\eta-x)]^{-1} [\sin(\pi\eta/2d) - \\ &- \sin(\pi x/2d)] d\eta = \\ &= -\pi \oint_L [\omega(\eta) (\eta-\tau)]^{-1} y^*(\eta, x) [\sin(\pi\eta/2d) - \sin(\pi x/2d)] d\eta + \\ &+ \pi^2 y^*(\tau, x) \end{aligned} \quad (A2:8)$$

Hence

$$I_{14} = (I_{12})_{\tau \rightarrow x} = -\pi \oint_L [\omega(\eta) (\eta-x)]^{-1} y^*(\eta, x) [\sin(\pi\eta/2d) - \sin(\pi x/2d)] d\eta + \pi^2 y'(x) \quad (A2:9)$$

Now from (A2:3) - (A2:9):

$$\begin{aligned} I_0 &= -I_1 - I_2 = -I_1 - (x-\tau)^{-1} [I_{11} - I_{12} - I_{13} + I_{14}] = \\ &= -\pi^2/2d \oint_L [\omega(\eta) (\eta-x) (\eta-\tau)]^{-1} y(\eta) \cos(\pi\eta/2d) d\eta + \\ &+ \pi \oint_L [\omega(\eta) (\eta-x) (\eta-\tau)]^{-1} [y^*(\eta, x) - y'(\eta)] [\sin(\pi\eta/2d) - \sin(\pi x/2d)] d\eta - \pi^2 (x-\tau)^{-1} G(\tau, x) \end{aligned}$$

where

$$\begin{aligned} G(\tau, x) &= -(\pi/4d) [\omega(\tau)]^{-2} [y(\tau) \sin(\pi\tau/d) - y(a) \sin(\pi a/d)] + \\ &+ (\pi/4d) [\omega(x)]^{-2} [y(x) \sin(\pi x/d) - y(a) \sin(\pi a/d)] + \\ &+ y'(\tau) - y^*(\tau, x) \end{aligned}$$

I was given by (A2:1)

$$\begin{aligned} I &= \oint_L \omega(\tau) T(\tau) [I_0 - 2\theta J_0] d\tau = \\ &= -(\pi^2/2d) \oint_L \omega(\tau) T(\tau) \oint_L [\omega(\eta) (\eta-x) (\eta-\tau)]^{-1} y(\eta) \cos(\pi\eta/2d) d\eta d\tau + \\ &+ \pi \oint_L \omega(\tau) T(\tau) \oint_L [\omega(\eta) (\eta-x) (\eta-\tau)]^{-1} [y^*(\eta, x) - y'(\eta)] [\sin(\pi\eta/2d) - \sin(\pi x/2d)] d\eta d\tau - \\ &- \sin(\pi x/2d) \oint_L \omega(\tau) T(\tau) G(\tau, x) d\tau - \\ &- 2\theta \pi^2 \oint_L \omega(\tau) T(\tau) y'^*(x, \tau) d\tau + \\ &+ 2\theta \pi \oint_L \omega(\tau) T(\tau) \oint_L [\omega(\eta) (\eta-\tau)]^{-1} y'^*(x, \eta) [\sin(\pi\eta/2d) - \sin(\pi x/2d)] d\eta d\tau \end{aligned} \quad (A2:10)$$

The last term can, after change of the order of integration and evaluation of integrals according to Appendix I, be reduced to:

$$2\theta\pi\oint_L[\omega(\eta)(\eta-x)]^{-1}y'(\eta)[\sin(\pi\eta/2d) - \sin(\pi x/2d)]\oint_L(\eta-\tau)^{-1}\omega(\tau)T(\tau)d\tau d\eta$$

After change of the order of integration in the first two terms of (A2:10), the final expression for I reads

$$\begin{aligned} I = & -(\pi^2/2d)\oint_L[\omega(\xi)(\xi-x)]^{-1}y(\xi)\cos(\pi\xi/2d)\oint_L(\xi-\eta)^{-1}\omega(\eta)T(\eta)d\eta d\xi + \\ & + (\pi^4/2d)y(x)\cos(\pi x/2d)T(x) + \\ & - \pi\oint_L[\omega(\xi)(\xi-x)]^{-1}y^*(\xi,x)[\sin(\pi\xi/2d) - \\ & - \sin(\pi x/2d)]\oint_L(\xi-\eta)^{-1}\omega(\eta)T(\eta)d\eta d\xi - \\ & - (\pi^3/4d)\oint_L[\omega(\xi)(\xi-x)]^{-1}[y(\xi)\sin(\pi\xi/d) - y(a)\sin(\pi a/d)]T(\xi)d\xi + \\ & + \pi^2\oint_L(\xi-x)^{-1}\omega(\xi)y'(\xi)T(\xi)d\xi - \\ & - \pi^2\oint_L(\xi-x)^{-1}\omega(\xi)y^*(\xi,x)T(\xi)d\xi + \\ & + (\pi^3/4d)[\omega(x)]^{-2}[y(x)\sin(\pi x/d) - y(a)\sin(\pi a/d)]\oint_L(\xi-x)^{-1}\omega(\xi)T(\xi)d\xi - \\ & - 2\theta\pi^2\oint_L\omega(\xi)T(\xi)y'^*(\xi,x)d\xi - \\ & - (1-2\theta)\pi\oint_L[\omega(\xi)(\xi-x)]^{-1}y'(\xi)[\sin(\pi\xi/2d) - \\ & - \sin(\pi x/2d)]\oint_L(\xi-\eta)^{-1}\omega(\eta)T(\eta)d\eta d\xi \end{aligned}$$

The expressions for $D_I(x)$ and $D_{II}(x)$ then read

$$\begin{aligned} \pi^2\omega(x)D_{I,II}(x) = & \oint_L(\xi-x)^{-1}\omega(\xi)T_{n,t}(\xi)d\xi + \\ & + (1/2d)\oint_L[\omega(\xi)(\xi-x)]^{-1}y(\xi)\cos(\pi\xi/2d)\oint_L(\xi-\eta)^{-1}\omega(\eta)T_{t,n}(\eta)d\eta d\xi - \end{aligned}$$

$$\begin{aligned}
& - (\pi^2/2d) y(x) \cos(\pi x/2d) T_{t,n}(x) - \\
& - (1/\pi) \oint_L [\omega(\xi) (\xi-x)]^{-1} y^*(\xi, x) [\sin(\pi \xi/2d) - \\
& - \sin(\pi x/2d)] \oint_L (\xi-\eta) \omega(\eta) T_{t,n}(\eta) d\eta d\xi + \\
& + (\pi/4d) \oint_L [\omega(\xi) (\xi-x)]^{-1} [y(\xi) \sin(\pi \xi/d) - y(a) \sin(\pi a/d)] T_{t,n}(\xi) d\xi - \\
& - \oint_L (\xi-x)^{-1} \omega(\xi) y'(\xi) T_{t,n}(\xi) d\xi + \\
& + \oint_L (\xi-x)^{-1} \omega(\xi) y^*(\xi, x) T_{t,n}(\xi) d\xi - \tag{A2:11} \\
& - (\pi/4d) [\omega(x)]^{-2} [y(x) \sin(\pi x/d) - \\
& - y(a) \sin(\pi a/d)] \oint_L (\xi-x)^{-1} \omega(\xi) T_{t,n}(\xi) d\xi + \\
& + 2\theta \oint_L \omega(\xi) y'^*(\xi, x) T_t(\xi) d\xi + \\
& + (1/\pi) (1-2\theta) \oint_L [\omega(\xi) (\xi-x)]^{-1} y'(\xi) [\sin(\pi \xi/2d) - \\
& - \sin(\pi x/2d)] \oint_L (\xi-\eta)^{-1} \omega(\eta) T_{t,n}(\eta) d\eta d\xi
\end{aligned}$$

Now it is assumed that $T_{n,t}(x)$ to the first order in y' can be written

$$T_{n,t}(x) = T_{n,t}^{(0)} + T_{n,t}^{(1)} y'(x) \tag{A2:12}$$

where $T_{n,t}^{(0)}$ and $T_{n,t}^{(1)}$ are constants.

Substitutions of (A2:12) into (A2:11) gives, neglecting terms of second or higher order in y and y' :

$$\begin{aligned}
\pi^2 \omega(x) D_{I,II}(x) &= T_{n,t}^{(0)} \oint_L (\xi-x)^{-1} \omega(\xi) d\xi + \\
&+ T_{n,t}^{(1)} \oint_L (\xi-x)^{-1} \omega(\xi) y'(\xi) d\xi +
\end{aligned}$$

$$\begin{aligned}
& + (1/2d)T_{t,n}^{(0)} \oint_L [\omega(\xi)(\xi-x)]^{-1} y(\xi) \cos(\pi\xi/2d) \oint_L (\xi-\eta)^{-1} \omega(\eta) d\eta d\xi - \\
& - (\pi^2/2d)T_{t,n}^{(0)} y(x) \cos(\pi x/2d) - \\
& - (1/\pi)T_{t,n}^{(0)} \oint_L [\omega(\xi)(\xi-x)]^{-1} y^*(\xi, x) [\sin(\pi\xi/2d) - \\
& - \sin(\pi x/2d)] \oint_L (\xi-\eta)^{-1} \omega(\eta) d\eta d\xi + \quad (A2:13) \\
& + (\pi/4d)T_{t,n}^{(0)} \oint_L [\omega(\xi)(\xi-x)]^{-1} [y(\xi) \sin(\pi\xi/d) - y(a) \sin(\pi a/d)] d\xi - \\
& - T_{t,n}^{(0)} \oint_L (\xi-x)^{-1} \omega(\xi) y'(\xi) d\xi + T_{t,n}^{(0)} \oint_L (\xi-x)^{-1} \omega(\xi) y^*(\xi, x) d\xi - \\
& - (\pi/4d)T_{t,n}^{(0)} [\omega(x)]^{-2} [y(x) \sin(\pi x/d) - y(a) \sin(\pi a/d)] \oint_L (\xi-x)^{-1} \omega(\xi) d\xi + \\
& + 2\theta T_t^{(0)} \oint_L \omega(\xi) y'^*(\xi, x) d\xi + \\
& + (1/\pi)(1-2\theta)T_{t,n}^{(0)} \oint_L [\omega(\xi)(\xi-x)]^{-1} y'(\xi) [\sin(\pi\xi/2d) - \\
& - \sin(\pi x/2d)] \oint_L (\xi-\eta)^{-1} \omega(\eta) d\eta d\xi
\end{aligned}$$

After further simplifications of (A2:13) according to Appendix 1, one arrives at

$$\begin{aligned}
\pi^2 \omega(x) D_{I,II}(x) & = -\pi T_{n,t}^{(0)} \sin(\pi x/2d) + \\
& + [T_{n,t}^{(1)} + 2\theta T_t^{(0)}] \oint_L (\xi-x)^{-1} \omega(\xi) y'(\xi) d\xi + \\
& + (\pi/4d)T_{t,n}^{(0)} \oint_L [\omega(\xi)(\xi-x)]^{-1} y(\xi) \sin(\pi\xi/d) d\xi - \quad (A2:14) \\
& - T_{t,n}^{(0)} \oint_L [\omega(\xi)(\xi-x)]^{-1} y^*(\xi, x) [\sin(\pi\xi/2d) - \sin(\pi x/2d)] \sin(\pi\xi/2d) d\xi + \\
& + (\pi^2/4d)T_{t,n}^{(0)} [\omega(x)]^{-2} [y(x) \sin(\pi x/d) - y(a) \sin(\pi a/d)] \sin(\pi x/2d) + \\
& + 2\pi\theta T_t^{(0)} y'(x) \sin(\pi x/2d) + \\
& + (1-2\theta)T_{t,n}^{(0)} \oint_L [\omega(\xi)(\xi-x)]^{-1} y'(\xi) [\sin(\pi\xi/2d) - \sin(\pi x/2d)] \sin(\pi\xi/2d) d\xi + \\
& + T_{t,n}^{(0)} H(\xi, x)
\end{aligned}$$

where

$$\begin{aligned}
 H(\xi, x) = & (\pi/4d) \oint_L [\omega(\xi) (\xi-x)]^{-1} [y(\xi) \sin(\pi\xi/d) - y(a) \sin(\pi a/d)] d\xi - \\
 & - \oint_L (\xi-x)^{-1} \omega(\xi) d\xi + \oint_L (\xi-x)^{-1} \omega(\xi) y^*(\xi, x) d\xi - \\
 & - (\pi^2/2d) y(x) \cos(\pi x/2d)
 \end{aligned}$$

For further simplification of the expression for $D_{I,II}(x)$, the different terms of (A2:14) will be treated separately.

First consider

$$S_1 = \oint_L (\xi-x)^{-1} \omega(\xi) y'(\xi) d\xi$$

A change of variable, $\xi \rightarrow \xi - 2nd$, in S_1 gives

$$S_1 = \oint_{-a}^a \omega(\xi) y'(\xi) Z_1(\xi, x) d\xi$$

where

$$\begin{aligned}
 Z_1(\xi, x) = & \sum_{n=-\infty}^{\infty} (-1)^n (\xi-x-2nd)^{-1} = (\xi-x)^{-1} - \\
 & - [(\xi-x)/2d^2] \sum_{n=1}^{\infty} (-1)^n \left[n^2 - [(\xi-x)/2d]^2 \right]^{-1}
 \end{aligned}$$

Now

$$\sum_{n=1}^{\infty} (-1)^n (n^2 - \alpha^2)^{-1} = (1/2\alpha^2) - 2\pi\alpha \operatorname{cosec}(\pi\alpha) \quad (\text{A2:15})$$

Thus

$$Z_1(\xi, x) = (\pi/2d) \operatorname{cosec}[\pi(\xi-x)/2d]$$

and

$$S_1 = (\pi/2d) \oint_{-a}^a \left[\sin[\pi(\xi-x)/2d] \right]^{-1} \omega(\xi) y'(\xi) d\xi$$

The sum of the second and third terms is

$$\begin{aligned}
 S_2 &= (\pi/4d) \oint_L [\omega(\xi) (\xi-x)]^{-1} y(\xi) \sin(\pi\xi/d) d\xi - \\
 &- \oint_L [\omega(\xi) (\xi-x)]^{-1} y^*(\xi, x) [\sin(\pi\xi/2d) - \sin(\pi x/2d)] \sin(\pi\xi/2d) d\xi = \\
 &= \oint_L [\omega(\xi)]^{-1} y(\xi) \sin(\pi\xi/2d) A(\xi, x) d\xi + y(x) \oint_L B(\xi, x) d\xi = \\
 &= S_{21} + S_{22}
 \end{aligned}$$

where

$$\begin{aligned}
 A(\xi, x) &= (\xi-x)^{-1} \left[(\pi/2d) \cos(\pi\xi/2d) - (\xi-x)^{-1} [\sin(\pi\xi/2d) - \sin(\pi x/2d)] \right] \\
 B(\xi, x) &= [\omega(\xi) (\xi-x)^2]^{-1} [\sin(\pi\xi/2d) - \sin(\pi x/2d)] \sin(\pi\xi/2d)
 \end{aligned}$$

First study S_{21}

Possible singular point is $\xi = x$.

But

$$\lim_{\xi \rightarrow x} A(\xi, x) = -(\pi^2/4d^2) \sin(\pi x/2d)$$

which means that S_{21} is not singular.

A change of variable, $\xi \rightarrow \xi - 2nd$, gives

$$S_{21} = \oint_{-a}^a [\omega(\xi)]^{-1} y(\xi) \sin(\pi\xi/2d) Z_2(\xi, x) d\xi$$

where

$$\begin{aligned}
 Z_2(\xi, x) &= (\pi/2d) \cos(\pi\xi/2d) \sum_{n=-\infty}^{\infty} (-1)^n (\xi-x+2nd)^{-1} - \\
 &- \sum_{n=-\infty}^{\infty} (-1)^n (\xi-x+2nd)^{-2} [(-1)^n \sin(\pi\xi/2d) - \sin(\pi x/2d)]
 \end{aligned}$$

(A2:15) and

$$\sum_{n=-\infty}^{\infty} (n-\alpha)^{-2} = \pi^2 \operatorname{cosec}^2(\pi\alpha)$$

$$\sum_{n=-\infty}^{\infty} (-1)^n (n-\alpha)^{-2} = \pi^2 \cos(\pi\alpha) \operatorname{cosec}^2(\pi\alpha)$$

gives

$$\begin{aligned} z_2 = (\pi/2d)^2 \operatorname{cosec}[\pi(\xi-x)/2d] & \left\{ \cos(\pi\xi/2d) - \right. \\ & \left. - \operatorname{cosec}[\pi(\xi-x)/2d] \left[\sin(\pi\xi/2d) \cos[\pi(\xi-x)/2d] + \sin(\pi x/2d) \right] \right\} = 0 \end{aligned}$$

Now turn to S_{22} .

The integral over $B(\xi, x)$ can be calculated with the aid of Appendix 1:

$$\begin{aligned} \oint_L B(\xi, x) &= \\ &= \oint_L [\omega(\xi) (\xi-x)^2]^{-1} [\sin(\pi\xi/2d) - \sin(\pi x/2d)] \sin(\pi\xi/2d) d\xi = \\ &= \lim_{\eta \rightarrow x} \oint_L [\omega(\xi) (\xi-x) (\xi-\eta)]^{-1} \sin^2(\pi\xi/2d) d\xi - \\ &- \sin(\pi x/2d) \lim_{\eta \rightarrow x} \oint_L [\omega(\xi) (\xi-x) (\xi-\eta)]^{-1} \sin(\pi\xi/2d) d\xi = \\ &= C + D \end{aligned}$$

Now

$$[(\xi-x)(\xi-\eta)]^{-1} = (x-\eta)^{-1} [(\xi-x)^{-1} - (\xi-\eta)^{-1}]$$

which gives

$$C = (1/2) \lim_{\eta \rightarrow x} \left\{ (x-\eta)^{-1} \left[\oint_L [\omega(\xi) (\xi-x)]^{-1} d\xi - \oint_L [\omega(\xi) (\xi-\eta)]^{-1} d\xi - \right. \right.$$

$$\begin{aligned}
& - \oint_L [\omega(\xi) (\xi-x)]^{-1} \cos(\pi\xi/d) d\xi + \oint_L [\omega(\xi) (\xi-\eta)]^{-1} \cos(\pi\xi/d) d\xi \Big] \Big\} \\
& = \pi \lim_{\eta \rightarrow x} \left\{ (x-\eta)^{-1} [\sin(\pi x/2d) - \sin(\pi\eta/2d)] \right\} = \\
& = (\pi^2/2d) \cos(\pi x/2d)
\end{aligned}$$

$$D = 0$$

and

$$S_{22} = (\pi^2/2d) y(x) \cos(\pi x/2d)$$

Thus

$$S_2 = (\pi^2/2d) y(x) \cos(\pi x/2d)$$

Next consider

$$S_3 = \oint_L [\omega(\xi) (\xi-x)]^{-1} y'(\xi) [\sin(\pi\xi/2d) - \sin(\pi x/2d)] \sin(\pi\xi/2d) d\xi$$

The variable change $\xi \rightarrow \xi - 2nd$ gives

$$S_3 = \int_{-a}^a [\omega(\xi)]^{-1} y'(\xi) \sin(\pi\xi/2d) Z_3(\xi, x) d\xi$$

with

$$\begin{aligned}
Z_3(\xi, x) &= \sin(\pi\xi/2d) \sum_{n=-\infty}^{\infty} (-1)^n (\xi-x+2nd)^{-1} - \\
&\quad - \sin(\pi x/2d) \sum_{n=-\infty}^{\infty} (\xi-x+2nd)^{-1}
\end{aligned}$$

(A2:15) and the relation

$$\sum_{n=1}^{\infty} (n^2 - \alpha^2)^{-1} = (1/2\alpha^2) - (\pi/2\alpha) \cot(\pi\alpha)$$

give

$$Z_3(\xi, x) = (\pi/2d) \operatorname{cosec}[\pi(\xi-a)/2d] \left[\sin(\pi\xi/2d) - \right. \\ \left. - \sin(\pi x/2d) \cos[\pi(\xi-x)/2d] \right]$$

Then

$$S_3 = (\pi/2d) \int_{-a}^a \left[\omega(\xi) \sin[\pi(\xi-a)/2d] \right]^{-1} y'(\xi) \left[\sin(\pi\xi/2d) - \right. \\ \left. - \sin(\pi x/2d) \cos[\pi(\xi-x)/2d] \right] d\xi = \\ = (\pi/2d) \cos(\pi x/2d) \int_{-a}^a [\omega(\xi)]^{-1} y'(\xi) \sin(\pi\xi/2d) d\xi = 0$$

since the integrand is odd in ξ .

The last term is

$$H(\xi, x) = (\pi/4d) \oint_L [\omega(\xi)(\xi-x)]^{-1} [y(\xi) \sin(\pi\xi/d) - y(a) \sin(\pi a/d)] d\xi - \\ - \oint_L (\xi-x)^{-1} \omega(\xi) y'(\xi) d\xi + \oint_L (\xi-x)^{-1} \omega(\xi) y^*(\xi, x) d\xi - \\ - (\pi^2/2d) y(x) \cos(\pi x/2d)$$

In the first integral, according to Appendix 1

$$y(a) \sin(\pi a/d) \oint_L [\omega(\xi)(\xi-x)]^{-1} d\xi = 0$$

Now integrate the last integral by parts:

$$\oint_L (\xi-x)^{-1} \omega(\xi) y^*(\xi, x) d\xi = \left\{ -(\xi-x)^{-1} \omega(\xi) [y(\xi) - y(x)] \right\}_L + \\ + \oint_L (\xi-x)^{-1} \omega(\xi) y'(\xi) d\xi + \int_L (\xi-x)^{-1} \omega'(\xi) [y(\xi) - y(x)] d\xi = \\ = \oint_L (\xi-x)^{-1} \omega(\xi) y'(\xi) d\xi - (\pi/4d) \oint_L [\omega(\xi)(\xi-x)]^{-1} y(\xi) \sin(\pi\xi/d) d\xi + \\ + (\pi^2/2d) y(x) \cos(\pi x/2d)$$

since the expression inside brackets $\left\{ \right\}$ is zero and

$$\oint_L (\xi-x)^{-1} \omega'(\xi) d\xi = -(\pi^2/2d) \cos(\pi x/2d)$$

according to Appendix 1.

Thus

$$H(\xi, x) = 0$$

Finally the expressions for $D_I(x)$ and $D_{II}(x)$ are arrived at:

$$\begin{aligned} \pi^2 \omega(x) D_{I, II}(x) &= -\pi T_{n, t}^{(0)} \sin(\pi x/2d) + \\ &+ (\pi/2d) \left[T_{n, t}^{(1)} + 2\theta T_t^{(0)} \right] \int_{-a}^a \operatorname{cosec}[\pi(\xi-x)/2d] \omega(\xi) y'(\xi) d\xi + \\ &+ (\pi^2/2d) T_{t, n}^{(0)} y(x) \cos(\pi x/2d) + \\ &+ (\pi^2/4d) T_{t, n}^{(0)} [\omega(x)]^{-2} [y(x) \sin(\pi x/d) - y(a) \sin(\pi a/d)] \sin(\pi x/2d) + \\ &+ 2\pi \theta T_t^{(0)} y'(x) \sin(\pi x/2d) \end{aligned}$$

APPENDIX 3

Calculation of $K_I(a)$ and $K_{II}(a)$

$K_I(a)$ and $K_{II}(a)$ are given by the limiting values

$$K_{I,II}(a) = \pi(2\pi)^{\frac{1}{2}} \lim_{x \rightarrow a} [(a-x)^{\frac{1}{2}} D_{I,II}(x)]$$

With $D_I(x)$ and $D_{II}(x)$ according to (13):

$$\begin{aligned} K_{I,II}(a) = & (2/\pi)^{\frac{1}{2}} \lim_{x \rightarrow a} \left\{ [\omega(x)]^{-1} (a-x)^{\frac{1}{2}} \left[-\pi T_{n,t}^{(0)} \sin(\pi x/2d) + \right. \right. \\ & + (\pi/2d) [T_{n,t}^{(1)} + 2\theta T_t^{(0)}] \oint_{-a}^a \left[\sin[\pi(\xi-x)/2d] \right]^{-1} \omega(\xi) y'(\xi) d\xi + \\ & + (\pi^2/2d) T_{t,n}^{(0)} y(x) \cos(\pi x/2d) + \\ & + (\pi^2/4d) T_{t,n}^{(0)} [\omega(x)]^{-2} [y(x) \sin(\pi x/d) - y(a) \sin(\pi a/d)] \sin(\pi x/2d) + \\ & \left. \left. + 2\pi\theta T_t^{(0)} y'(x) \sin(\pi x/2d) \right] \right\} \end{aligned} \quad (A3:1)$$

Here the notation $\omega(x)$ in the integrals is used to mean $\omega_+(x)$.

First consider the factor

$$f = \lim_{x \rightarrow a} \left[(a-x)^{\frac{1}{2}} [\omega(x)]^{-1} \right]$$

where

$$\begin{aligned} \omega(x) &= [\sin^2(\pi a/2d) - \sin^2(\pi x/2d)]^{\frac{1}{2}} = \\ &= [\sin(\pi a/2d) - \sin(\pi x/2d)]^{\frac{1}{2}} [\sin(\pi a/2d) + \sin(\pi x/2d)]^{\frac{1}{2}} \\ &= f_1 \cdot f_2 \end{aligned}$$

$$\text{As } x \rightarrow a, \quad f_2 \rightarrow [2\sin(\pi a/2d)]^{\frac{1}{2}}$$

For f_1 , put $\epsilon = a-x$. Then, as $\epsilon/a \rightarrow 0$

$$\begin{aligned} f_1 &= [\sin(\pi a/2d) - \sin(\pi(a-\epsilon)/2d)]^{\frac{1}{2}} = \\ &= [\sin(\pi a/2d) - \sin(\pi a/2d)\cos(\pi\epsilon/2d) + \sin(\pi\epsilon/2d)\cos(\pi a/2d)]^{\frac{1}{2}} \rightarrow \\ &\rightarrow [(\pi/2d)\cos(\pi a/2d)]^{\frac{1}{2}}(a-x)^{\frac{1}{2}} \end{aligned}$$

Hence

$$\begin{aligned} \omega(x) &\rightarrow [(\pi/d)\sin(\pi a/2d)\cos(\pi a/2d)(a-x)]^{\frac{1}{2}} \\ &= [(\pi/2d)\sin(\pi a/d)(a-x)]^{\frac{1}{2}} \end{aligned} \quad (A3:2)$$

$$\text{Thus } f = [(\pi/2d)\sin(\pi a/d)]^{-\frac{1}{2}}$$

Next consider

$$g = \lim_{x \rightarrow a} [\omega(x)]^{-2} [y(x)\sin(\pi x/d) - y(a)\sin(\pi a/d)]\sin(\pi x/2d)$$

The same substitution as for f_1 , $\epsilon = a-x$, gives

$$g = -(2d/\pi)y'(a)\sin(\pi a/2d) - y(a)\cos(\pi a/d)\sec(\pi a/2d)$$

The integral can be rewritten as

$$\begin{aligned} R &= \lim_{x \rightarrow a} \oint_{-a}^a \left[\sin[\pi(\xi-x)/2d] \right]^{-1} \omega(\xi)y'(\xi)d\xi = \\ &= \oint_{-a}^a \left[\sin[\pi(\xi-a)/2d] \right]^{-1} \omega(\xi)y'(\xi)d\xi = \\ &= -\cos(\pi a/2d) \int_{-a}^a [\omega(\xi)]^{-1}y'(\xi)\sin(\pi\xi/2d)d\xi - \\ &\quad - \sin(\pi a/2d) \int_{-a}^a [\omega(\xi)]^{-1}y'(\xi)\cos(\pi\xi/2d)d\xi \end{aligned}$$

The first integral is zero as the integrand is odd.

Thus

$$R = -\sin(\pi a/2d) \int_{-a}^a [\omega(\xi)]^{-1}y'(\xi)\cos(\pi\xi/2d)d\xi$$

Thus the expressions for $K_{I,II}(a)$ are given by

$$\begin{aligned}
 K_{I,II}(a) = & -T_{t,n}^{(0)} [d/\sin(\pi a/d)]^{\frac{1}{2}} \sin(\pi a/2d) \left[y'(a) (1-4\theta) + \right. \\
 & + 2 \left(T_{n,t}^{(0)} / T_{t,n}^{(0)} \right) - (\pi/d) y(a) \operatorname{cosec}(\pi a/d) + \\
 & \left. + d^{-1} \left(T_{n,t}^{(1)} + 2\theta T_t^{(0)} \right) / (T_{t,n}^{(0)}) \int_{-a}^a [\omega(\xi)]^{-1} y'(\xi) \cos(\pi \xi/2d) d\xi \right]
 \end{aligned}$$

